XFEM analysis of fiber bridging in mixed-mode crack propagation in composites

A. Afshar, A. Daneshyar, S. Mohammadi *

School of Civil Engineering, College of Engineering, University of Tehran, Tehran, Iran

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Arbitrary crack propagation

A B S T R A C T

In this work, the extended finite element method (XFEM) is further generalized to study the fiber bridging phenomenon in fracture analysis of unidirectional composites. Treating composites as a biphasic material, fibers effects on increasing fracture toughness are modeled by non-linear springs integrated in the XFEM formulation of displacement discontinuity. Combining the capability of XFEM in modeling arbitrary crack path with the traction-separation representation of the fracture process zone in the crack-bridged zone model, this formulation allows for simulation of fiber bridging in a crack propagation process without a priori knowledge of the crack path. Moreover, the criterion originally proposed for predicting the crack propagation path in orthotropic solids is modified to incorporate the effects of fiber orientation. Several numerical simulations of mode I and mixed-mode problems are provided, and the corresponding force–displacement curves are analyzed to address the accuracy and efficiency of the proposed method.

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1. Introduction

Reinforcing a matrix with glass or carbon fibers not only improves the stiffness, but also increases its crack growth resistance. When a crack starts to propagate in a fiber reinforced composite, the fibers bridge the crack and exert force on the crack face, enhancing the fracture toughness of the composite. Due to the increasing significance of composite materials in aerospace and automotive industry, many researchers tried to investigate the effects of fiber bridging on crack propagation in composites by experimental and/or numerical methods. The aim of this work is to propose a simple yet powerful formulation for simulation of fiber bridging in arbitrary crack propagation in composites.

In unidirectional fiber reinforced composites, many researchers have tried to incorporate the effects of fiber cross-over bridging into a cohesive zone model by utilizing a traction-separation law of the fracture process zone [1–4]. Adopting appropriate forms of the traction-separation law and accurate calibration of cohesive parameters have allowed these works to simulate the fiber bridging in the fracture process zone. Nonetheless, these studies are restricted to discrete interelement approach of the cohesive zone model [5–8], in which the crack growth path must be known a

priori. While this assumption appears to be reasonable for interlaminar fracture, the crack growth path may not be predetermined in a general mixed mode fracture problem of composites, and hence a more realistic method is required to allow for arbitrary crack propagation path. Recently, Rudraraju et al. [9–11] employed a micro–macro decomposition of the displacement field, combined with the embedded element discontinuity, to investigate fracture of composite laminates, focusing on determining force–displacement curves and analysis of the traction-separation law and the bridging zone. While the embedded discontinuity approach of the cohesive zone model [12,13] can be utilized for fracture analysis of composites [14,15], its numerical implementation results in an asymmetric stiffness matrix, nonconforming displacement kinematics [16] and may cause stability issues [17]. Therefore, an alternative formulation based on the extended finite element method (XFEM) is proposed in this study to account for the effects of fiber bridging in general arbitrary crack path problems.

XFEM, originally proposed by Belytschko and Black [18], is a powerful numerical technique suited to problems concerning discontinuities, in which neither singular elements are required to reproduce the crack tip singularity nor element edges should conform to the crack face in a general crack propagation problem. For fracture analysis of composites, XFEM has been successfully used in both static and dynamic problems [19–27]. However, almost all of these studies are based on a linear elastic fracture mechanics, neglecting the nonlinearity effects of fiber bridging in the fracture process zone. Such a major drawback is surmounted in this work

* Corresponding author at: High Performance Computing Lab, School of Civil Engineering, University of Tehran, Tehran, Iran. Tel.: +98 21 6111 2258; fax: +98 21 6640 3808.

E-mail address: smoham@ut.ac.ir (S. Mohammadi).

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by simulating the fiber bridging, while preserving the framework of XFEM and its advantages.

In addition to the cohesive zone models, in which the material is considered to be homogenous, bridged crack models are also used to describe the nonlinear fracture process zone [28–37]. Bridged crack models are based on the assumption that the material is multiphase, and hence more than one phenomenon contributes to the fracture [38]. Separating the matrix phase failure from the fibers pull-out, the bridged crack models seem more promising for analysis of fracture in composites [39], and therefore these models are adopted in this study. Though very similar in the formulation, this distinction between the two models results in a removal of the stress singularity in the cohesive zone approach and a non-vanishing singular field in the bridged crack models. In the latter model, the crack growth criterion is mainly affected and a non-vanishing singular field in the bridged crack models.

Usually, the effects of fiber bridging in increasing the crack growth resistance is illustrated by a fracture resistance curve (R-curve). In a crack growth process which includes fiber bridging, the R-curve starts at the fracture initiation toughness, and then increases to a plateau. It has been shown that this curve is not a composite inherent characteristic and depends on the geometry of the specimen [40]. As a result, bridging laws should be used as the material property representing the fracture resistance. By using the bridging laws and the J-integral, many authors studied the fiber bridging phenomenon [41–43], and the same approach is considered here to construct the fracture resistance curve. More details are presented in Section 3.

The structure of the paper is as follows: Section 2 provides the concepts of the LEFM based XFEM formulation of the problem and how it can be employed to model crack propagation in brittle materials. Section 3 presents the basics of the bridging law, the softening behavior, and evaluation of the fracture toughness for bridged cracks. Numerical energy of energy release rate for determining the crack growth length and direction is described in Section 4. Then, in Section 5 the original formulation of XFEM is altered in order to account for the nonlinearity of the fiber bridging phenomenon. Section 6 verifies this formulation by solving mode I delamination problems and a number of mixed mode fracture simulations. Finally, Section 7 completes this paper by discussing the advantages of this formulation over the conventional XFEM and cohesive zone models.

2. XFEM for brittle fracture

2.1. Displacement approximation

Consider a crack in a fiber reinforced composite, as depicted in Fig. 1. Neglecting the effects of fiber bridging, the displacement approximation of the conventional finite element method is enriched by

\[
\mathbf{u}_{\text{XFEM}} = \mathbf{u}_{\text{FEM}} + \mathbf{u}_{\text{crack-face}} + \mathbf{u}_{\text{crack-tip}}
\]  

(1)

where \(\mathbf{u}_{\text{crack-face}}\) and \(\mathbf{u}_{\text{crack-tip}}\) are added in order to model the discontinuity in the displacement field and the singularity in the stress field. The expression for the discontinuity enriched part of the displacement field is [44]

\[
\mathbf{u}_{\text{crack-face}} = \sum_{i \in \mathcal{M}} N_i(x) \mathbf{H}(x) \mathbf{\alpha}_i
\]  

(2)

where \(\mathcal{M}\) is the set of nodes enriched by the Heaviside function. The Heaviside function for a point \(x\) is defined as [44]

\[
\mathbf{H}(x) = \begin{cases} 
+1 & \text{if } (x - x') \mathbf{e}_n > 0 \\
-1 & \text{otherwise} 
\end{cases}
\]  

(3)

where \(x'\) denotes the coordinate of the closest point of the crack to the point \(x\) (Fig. 2). \(\mathbf{\alpha}_i\) represents additional degrees of freedom used for modeling crack face discontinuity. In a similar manner, the expression for the crack tip enriched part of the displacement field is

\[
\mathbf{u}_{\text{crack-tip}} = \sum_{i \in \mathcal{N}} N_i(x) \left( \sum_{k \in \mathcal{F}} f_k(x) \mathbf{d}_k \right)
\]  

(4)

where \(\mathcal{N}\) is the set of nodes enriched by the crack tip functions, and \(\mathcal{F}\) is the set of \(f_k\) crack tip functions. Due to the presence of fibers, the following orthotropic functions are adopted [22]

\[
F(r, \theta) = \left\{ \sqrt{r} \cos \left( \frac{\theta_1}{2} \right) \sqrt{g_1(\theta)}, \sqrt{r} \cos \left( \frac{\theta_2}{2} \right) \sqrt{g_2(\theta)} \right\}
\]  

(5)

Fig. 2. Definition of the distance function and the equivalent domain integral.
with
\[ g_j(\theta) = \sqrt{(\cos(\theta) + \zeta_j \sin(\theta))^2 + (\beta_j \sin(\theta))^2} \quad (j = 1, 2) \] (6)
\[ \theta_k(\delta) = \tan^{-1}\left( \frac{\beta_k \sin(\delta)}{\cos(\delta) + \zeta_k \sin(\delta)} \right) \quad (k = 1, 2) \] (7)
where \( \zeta_j \) and \( \beta_j \) are the real and imaginary parts of the roots of the characteristic equation \([45]\)
\[ a_{11}^{(p)} \mu^4 - 2a_{12}^{(p)} \mu^3 + \left( 2a_{12}^{(p)} + a_{66}^{(p)} \right) \mu^2 - 2a_{22}^{(p)} \mu + a_{22}^{(p)} = 0 \] (8)
\[ \mu_1 = \zeta_1 + i\beta_1 \]
\[ \mu_2 = \zeta_2 + i\beta_2 \]
where \( a_{ij} \) are the elements of the compliance tensor \( a \)
\[ \epsilon = a \sigma \] (10)

More details of the XFEM implementation are presented in Appendix A.

2.2. Updating degrees of freedom

Due to the movement of the crack tip position, enriched nodes should be updated accordingly. In each crack propagation step, enrichments associated with the previous tip nodes are changed to the Heaviside type enrichment. As depicted in Fig. 3, elimination of previous degrees of freedom results in invalid discontinuity in the crack geometry which generates an out of balance force in the equilibrium equation. Nevertheless, by solving the equilibrium equation in the updated configuration, the inconsistency in the crack discontinuity geometry relaxes, and therefore this residual force vanishes. The detailed algorithm of this procedure in the \( n^{th} \) loading increment (Fig. 3) is provided below:

- Solve \( \mathbf{f}^{int} - \mathbf{f}^{int} = \mathbf{r} \) in configuration (a) using the iterative Newton–Raphson algorithm
- Check the propagation criterion \( G = G_{cr} \)
- Update the crack geometry and advance to configuration (b)
- Calculate \( \mathbf{u}, \mathbf{K} \) and \( \mathbf{f}^{int} \)
- Imbalance equation \( \mathbf{f}^{int} - \mathbf{f}_{int}^{int} = \mathbf{r} \neq \mathbf{0} \)

3. Fiber bridging concept

3.1. Bridging laws

Every model of fiber bridging requires a traction-separation law, describing the forces fibers exert on the crack faces. They usually consist of a linearly increasing line that reaches a maximum traction, and a softening part which could be linear or nonlinear.

Since a bi-linear softening law has been proved to provide acceptable results \([46]\), the same laws are employed in this study to describe the formulation of spring elements. In addition, tri-linear laws (Fig. 4 for mode I and Fig. 5 for mode II) are also utilized in this work since these models are proved to improve the accuracy of the simulation by incorporating the initial fracture toughness \([41,43,47]\).

3.2. Fracture resistance (the R-curve)

As mentioned earlier, for two failure phenomena are involved in building the R-curve for bridged crack models. A part of the fracture resistance is provided by the matrix toughness \( G_{crack-tip} \), and another part results from the traction forces of the fibers \( G_{fiber-bridging} \)
\[ G_R = G_{crack-tip} + G_{fiber-bridging} \quad (15) \]

with
\[ G_{fiber-bridging} = \int_0^\delta t(\delta') d\delta' \] (16)

where \( \delta \) is the opening of crack face in the position of the fibers. The first term in Eq. (15), \( G_{crack-tip} \), is the inherent fracture toughness of the matrix, and the second term is the energy dissipated in the fibers. While the matrix fracture toughness is constant, the dissipated energy in the fibers depends on the elongation \( \delta \) of the fibers bridging the crack. The maximum energy absorbed in the fibers

![Fig. 3. Updating the enriched nodes and correction of crack discontinuity after relaxation.](image)
equals the area under the traction-separation curve \( G_{IC} \) and \( G_{IC} \) in Figs. 4 and 5). By calculating the area under the traction-separation law for fibers that bridge the crack and adding this value to the matrix toughness, the fracture resistance curve can be estimated.

4. Crack propagation scheme

4.1. Basic algorithm

An energy based crack propagation algorithm is adopted in this work. In each step of the loading process, the crack propagates incrementally with small segments, until its energy release rate drops below the matrix fracture toughness. Afterward, the next loading step is applied, and this process repeats until the final failure of the specimen. In this manner, the corresponding load–displacement and fracture resistance curves of the specimen are obtained. This algorithm further requires a method for calculating the energy release rate to evaluate the crack growth length in each step of the loading, and a criterion for determination of the propagation direction.

4.2. Crack growth length

The energy release rate of the crack during the growth is required for estimation of the growth length. The energy release rate could be quantified using the path-independent contour integral. In this work, the equivalent domain integral is utilized to calculate the \( J \)-integral

\[
J = \int_A (\sigma_y u_{y1} - w \delta_{ij}) q_j dA
\]  

(17)

In the above integral, \( \delta_{ij} \) is the Kronecker delta and \( w \) is the strain energy density, defined as

\[
w = \frac{1}{2} (\sigma_{11} e_{11} + \sigma_{22} e_{22} + 2 \sigma_{12} e_{12})
\]  

(18)

In Eq. (17), \( A \) is the interior region of an arbitrary contour \( \Gamma \) enclosing the crack tip (Fig. 2), and \( q \) is a smooth function, which equals unity on the interior boundary of \( A \) and zero on the exterior boundary, as depicted in Fig. 2.

4.3. Crack propagation direction

In the numerical simulations the crack is deemed to propagate in the direction \( \theta \) at which the circumferential stress is maximum. Therefore, the criterion originally proposed by Saouma et al. [48] is modified by contributing the angle of fiber directions,

\[
\begin{align*}
\text{Max} & \left( \frac{\Re \{ A (B \mu_1 - C \mu_2 ) \} + \frac{\mu_2}{K_1} \Re \{ A (B - C) \} } {\cos^2 (\theta + \alpha) + \frac{\mu_2}{K_2} \sin^2 (\theta + \alpha)} \right) \\
A & = \frac{1}{\mu_1 - \mu_2} \\
B & = (\mu_2 \sin (\theta - \theta_{mat} + \alpha) + \cos (\theta - \theta_{mat} + \alpha))^2 \\
C & = (\mu_1 \sin (\theta - \theta_{mat} + \alpha) + \cos (\theta - \theta_{mat} + \alpha))^2 \\
M & = \int_A \left( \sigma_y u_{y1}^{\text{max}} + \sigma_y u_{y1} - \frac{1}{2} (\sigma_{ik} e_{ik}^{\text{max}} + \sigma_{ik} e_{ik}) \delta_{ij} \right) q_j dA
\end{align*}
\]  

(19)

(20)

(21)

(22)

(23)

where \( \theta_{mat} \) and \( \alpha \) are the fibers direction and the crack angle, respectively. \( \mu_i \) are the roots of the characteristic Eq. (8). \( K_1 \) and \( K_2 \) are mode I and II stress intensity factors, extracted from the following interaction integral,

\[
M = \int_A \left( \sigma_y u_{y1}^{\text{max}} + \sigma_y u_{y1} - \frac{1}{2} (\sigma_{ik} e_{ik}^{\text{max}} + \sigma_{ik} e_{ik}) \delta_{ij} \right) q_j dA
\]  

(23)

To numerically evaluate the domain form of the interaction integral, the same finite element mesh is employed to calculate different terms of Eq. (23)(see Fig. 6). The standard \( 4 \times 4 \) Gauss rule is used for ordinary elements, while the subdomain triangulation technique is employed for enriched elements. For the auxiliary displacement and stress fields, the asymptotic solution of a crack in an orthotropic medium is employed [44] (see Appendix B).
The real mode I and II stress intensity factors can then be determined by solving the following simultaneous equations obtained from two different assumptions for the auxiliary field,

\[ M_l^1 = 2c_{11}(K_1^a + 1) \quad (K_1^a = 1 \text{ and } K_1^a = 0) \]
\[ M_l^2 = c_{12}K_1^a + 2c_{22}K_2^a \quad (K_1^a = 0 \text{ and } K_1^a = 1) \]

with

\[ c_{11} = -\frac{\alpha_{12}}{2} \text{Im}(\mu_1 + \mu_2) \]
\[ c_{12} = -\frac{\alpha_{22}}{2} \text{Im}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right) + \frac{\alpha_{11}}{2} \text{Im}(\mu_1, \mu_2) \]
\[ c_{22} = -c_{11} \text{Im}(\mu_1 + \mu_2) \]

where are the elements of the compliance tensor \( \mathbf{a} \) (Eq. (10)).

Following the calculation of \( K_1 \) and \( K_2 \), the crack growth direction \( \theta \) can be obtained by maximizing Eq. (19), which depends to \( K_1 \) and \( K_2 \), a number of material dependent constants and the relative angles of long fibers and the crack.

As illustrated in the numerical examples, criterion (19) ensures that the propagation direction is accurately affected by both the fibers direction and the specimen configuration.

### 5. Modeling of the fiber bridging zone

In order to simulate the effects of fiber bridging, spring elements are placed between a number of points located at different sides of the crack in a Heaviside element (Fig. 7). Displacements of these points, denoted by \( u^+ \) and \( u^- \), can be written as

\[ u^+ = \sum_i (N_i \hat{u}_i + N_i(1-H_i)\hat{a}_i) \]
\[ u^- = \sum_i (N_i \hat{u}_i + N_i(-1-H_i)\hat{a}_i) \]

where \( \hat{u} \) and \( \hat{a} \) are the real and enriched displacements of the Heaviside element, respectively.

Projections of these values onto the spring direction are

\[ u^+_s = \sum_i (N_i \hat{u}_i + N_i(1-H_i)\hat{a}_i) \hat{e}_i \]
\[ u^-_s = \sum_i (N_i \hat{u}_i + N_i(-1-H_i)\hat{a}_i) \hat{e}_i \]

The displacement of the spring element \( u^{\text{bar}} \) is approximated by the finite element shape functions

\[ u^{\text{bar}} = \sum_j N_j^{\text{bar}} \hat{u}_j^{\text{bar}} \]

where \( N_j^{\text{bar}} \) is the shape function of the spring element, \( \hat{u}_j^{\text{bar}} \) is the nodal displacements of the spring, and the vector containing these values are

\[ \hat{u}^{\text{bar}} = \left[ \begin{array}{c} \hat{u}_1^{\text{bar}} \\ \hat{u}_2^{\text{bar}} \end{array} \right] = \left[ \begin{array}{c} u^+_s \\ u^-_s \end{array} \right] \]

The corresponding strain of the spring element is computed from

\[ \varepsilon^{\text{bar}} = \frac{\partial N_j^{\text{bar}}}{\partial \xi} = \mathbf{B}^{\text{bar}} \hat{u}^{\text{bar}} \]

where \( \xi \) is the local axis of the spring, and \( \mathbf{B}^{\text{bar}} \) is the strain–displacement matrix for the spring element

\[ \mathbf{B}^{\text{bar}} = \frac{\partial N_j^{\text{bar}}}{\partial \xi} = \left[ \begin{array}{cc} -1 & 1 \end{array} \right] \]
Replacing Eq. (31) and Eq. (33) into Eq. (32) results in the following equations

\[ e_{\text{bar}} = 2 \sum_i (N_i \bar{\varepsilon}_s) \varepsilon_s \]  

(34)

\[ B^i = [2N_i \bar{\varepsilon}_s \bar{\varepsilon}_x, 2N_i \bar{\varepsilon}_s, \bar{\varepsilon}_y] \]  

(35)

\[ K^i = \int_{\Omega} ((B^i)^T \zeta_i B^i) d\Omega \]  

(36)

where \( B^i \) and \( K^i \) are the shape functions gradient and stiffness matrix of the spring element in the global coordinates.

Employing the Newton–Raphson iterative solver to determine the stiffness of the springs ensures that the solutions converge rapidly in all simulations, even with a high number of springs operating at the same time, and no numerical instability is observed. Furthermore, it should be noted that all the extra terms are lumped into the XFEM part of the stiffness matrix (Eqs. (35) and (36)). Therefore, once this formulation is incorporated into the stiffness matrix of the Heaviside elements, unlike the conventional cohesive zone models or models solely based on nonlinear springs, the fiber bridging effects are automatically simulated without a prior knowledge of the crack propagation path. For a mixed mode problem, the same formulation can be applied to model the mode II resistance of the fiber bridging. The efficacy of this formulation is examined by solving several problems in the next section.

6. Numerical examples

6.1. Mode I double cantilever beam

As the first example and in order to validate the proposed formulation, the standard double cantilever beam, which has been widely analyzed by experimental or numerical approaches, is reconsidered. The configuration is illustrated in Fig. 8 and the material properties are [49]

\[ E_1 = 130 \text{ GPa}; \quad E_2 = E_3 = 8.2 \text{ GPa} \]

\[ v_{12} = v_{13} = v_{23} = 0.3 \]

\[ G_{12} = 4.1 \text{ GPa} \]

The tri-linear form of the force-separation law, originally proposed by Ref. [49], with the properties presented in Table 1 is assumed.

### Table 1

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_0 ) (MPa)</td>
<td>45</td>
</tr>
<tr>
<td>( P_1 ) (MPa)</td>
<td>1</td>
</tr>
<tr>
<td>( d_0 ) (mm)</td>
<td>7.1 \times 10^{-4}</td>
</tr>
<tr>
<td>( d_1 ) (mm)</td>
<td>7.1 \times 10^{-3}</td>
</tr>
<tr>
<td>( d_2 ) (mm)</td>
<td>1.28</td>
</tr>
<tr>
<td>( G_{\text{crack-tip}} ) (N/mm)</td>
<td>0.2</td>
</tr>
</tbody>
</table>

The analysis is performed in the plane stress state and with the displacement control conditions using 2250 elements. The corresponding force-end beam displacement, presented in Fig. 9, shows a good agreement with the reference results. In the first stage of the loading, the response is linear, and when the fracture energy reaches the initial fracture toughness, the crack starts to propagate, causing a decrease in the stiffness of the specimen. Due to the fiber bridging effects, the loading capacity increases to the maximum value of 65N, and when the displacement of the fibers in the fracture process zone reaches its maximum, the loading capacity drops as a result of the final failure of the specimen. Moreover, the effects of fiber bridging mechanism in increasing the load carrying capacity is illustrated in Fig. 9 by modeling the same problem without considering the effects of fiber bridging. The corresponding R-curve for the current problem is depicted in Fig. 10, indicating the increasing trend in the fracture energy until reaching a plateau. In the bridging zone behind the propagating crack tip, fibers exert forces on the crack and increase the fracture toughness. The bridging length extends until the fracture process zone is fully developed. In the current problem, this length is approximately 19 mm. The maximum value of 1000 N/mm equals the total fracture resistance of the matrix and fibers bridging. Clearly, the results are in good agreement with the reference values [49].

6.2. Eccentric three-point bending specimen

An attempt is now made to extend the mode I crack propagation simulation to a problem with curved crack propagation. However, since no numerical simulation of a fiber bridged crack with arbitrary propagation path in the is available literature, the present
verification is applied to a specimen with material properties (cohesive strength and fracture toughness) that are irrelevant to conventional fiber reinforced composites. In fact, these material properties are more suited to cement-based composites instead of fiber reinforced polymers. With this fact in mind, an isotropic three-point bending specimen with a 0.4 (mm) eccentric crack is assumed (Fig. 11), and the materials and bi-linear spring properties are

\[ E = 100 \text{ MPa}; \quad \nu = 0 \]

Cohesive Strength \( (P_0) = 0.5 \text{ N/mm}^2 \)

Fracture Toughness \( (G_c) = 0.01 \text{ N/mm} \)

This problem was originally analyzed by Mergheim et al. [50] with the finite element method and a discrete damage-type model, and Sun et al. [51] by a cohesive zone model within a mesh-free method. The proposed formulation, combined with the criterion (9), is utilized to simulate the crack propagation, and to obtain the load–displacement curve, as depicted in Fig. 12. The same
trend of Fig. 9 in example 5.1 is also observed here. Numerical deformed configurations, the crack mouth opening, and the crack path in each stage of the applied displacement are depicted in Fig. 13.

Considering the fact that the behavior of springs includes a softening part, a mesh sensitivity analysis of the results seems necessary. Therefore, the same problem is analyzed by three different meshes, structured and unstructured. Fig. 14 shows that in all curves, even with a relatively coarse structured mesh, the results remain very close and accurate, clearly indicating that the proposed formulation does not suffer from any mesh dependency of the results.

### 6.3. Mixed-mode orthotropic composite

The proposed approach is now examined for the fracture analysis of composites with different fibers orientations (Fig. 15). Both mode-I and mode-II contributions of the fiber bridging are considered by assuming bi-linear forms of traction-separation law, with parameters presented in Tables 2 and 3, and the problem is solved in the plane stress state with 4714 elements. The configuration and material properties are

\[ E_{11} = 114.8 \text{ GPa}; \quad E_{22} = 11.7 \text{ GPa} \]

\[ G_{12} = 9.66 \text{ GPa}; \quad v_{12} = 0.21 \]

---

![Fig. 13. Numerical deformed configurations in different stages of applied displacements.](image1.png)

![Fig. 14. Mesh insensitivity of the results.](image2.png)
The corresponding force–displacement curves for various fiber angles are depicted in Fig. 16. It is observed that in configurations with lower fiber angles, the mode-II fracture mechanism becomes more significant, which provides more resistance to the crack growth, and increases the load-carrying capacity.

In order to find the crack propagation path, the modified maximum hoop stress criterion (Eq. (19)) is employed. According to Fig. 17, the crack path in this problem is mainly determined by the fiber orientation, rather than the loading/geometric configuration. Had we used a criterion excluding the effects of fiber orientations, the crack would have propagated towards the concentrated load. It should be noted, however, the fibers angle should be explicitly included in the propagation criterion, otherwise the propagation direction is incorrectly estimated. For cases where the fiber breakage is unlikely to occur (due to higher fracture resistance of fibers), it is expected that the crack should propagate in a direction generally parallel to fibers orientation. Such a propagation path is also affected by the loading and boundary condition. The original criterion of Saouma [48] for an orthotropic medium, nonetheless, does not incorporate the fibers orientation. To better examine the inaccuracy of the original criterion, this example is analyzed for two cases of 30° and 45° fibers with similar loading and boundary conditions. According to Fig. 18, the predicted propagation path for

![Fig. 15. Eccentric three-point bending composite specimen.](image1)

**Fig. 15.** Eccentric three-point bending composite specimen.

![Fig. 16. Load –displacement curve.](image2)

**Fig. 16.** Load –displacement curve.

![Fig. 17. Prediction of crack paths for different fiber orientations.](image3)

**Fig. 17.** Prediction of crack paths for different fiber orientations.

![Fig. 18. Inaccurate prediction of the propagation direction by the original criterion of Saouma [48].](image4)

**Fig. 18.** Inaccurate prediction of the propagation direction by the original criterion of Saouma [48].

### Table 2
Mode-I force-separation parameters.

<table>
<thead>
<tr>
<th>P₀ (MPa)</th>
<th>d₀ (mm)</th>
<th>d₁ (mm)</th>
<th>G_crack-tip (N/mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>0.000223</td>
<td>0.0223</td>
<td>0.67</td>
</tr>
</tbody>
</table>

### Table 3
Mode-II force-separation parameters.

<table>
<thead>
<tr>
<th>P₀ (MPa)</th>
<th>d₀ (mm)</th>
<th>d₁ (mm)</th>
<th>G_crack-tip (N/mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>0.000371</td>
<td>0.0371</td>
<td>1.67</td>
</tr>
</tbody>
</table>
the 30° lamina is located above the lamina with 45° fibers, clearly contradicting the expectation that the crack tends to propagate nearly parallel to fibers orientation. To improve the predictions, a slightly modified criterion has been proposed, which explicitly incorporates the fibers orientations. As a result, instead of having the absolute value of the crack angle alone, the relative angle of the crack and fibers orientation becomes decisive. The predictions for crack paths based on the modified criterion are now physically feasible and better consistent with high fiber fracture resistance (Fig. 17).

6.4. Fiber reinforced plate with three holes

The final example consists of a crack in a fiber reinforced plate with three holes (Fig. 19). The mechanical properties of the plate

![Fiber reinforced plate with three holes](image1)

**Fig. 19.** Fiber reinforced plate with three holes.

![Comparison of predicted crack propagation paths for different fiber orientations](image2)

**Fig. 20.** Comparison of predicted crack propagation paths for different fiber orientations.

![Comparison of force–displacement curves for different fiber orientations](image3)

**Fig. 21.** Comparison of force–displacement curves for different fiber orientations.
and the traction-separation parameters of the springs are presented in Tables 4–6, respectively. The analysis is performed in the plane strain and displacement control state with 4877 elements for different fibers orientations. It should also be noted that the same XFEM mesh has been used for all fibers orientations, irrespective of the crack propagation path (Fig. 19).

Fig. 20 illustrates the predicted crack propagation paths for different fiber orientations based on the modified criterion (19), which accounts for the collective effects of geometry, boundary condition and fiber orientations. Clearly, the crack tends to grow in a direction parallel with fibers, where much less fracture resistance is encountered.

Moreover, the force–displacement curves for all fiber orientations are depicted in Fig. 21. Again, in configurations in which the mode II fracture becomes more pronounced the load-carrying capacity of the specimen increases significantly. Also, Fig. 22 presents the stress field and numerical deformed configuration for the case of \( \theta_{mat} = 90^\circ \) and the applied displacement of \( \delta = 0.1 \) mm.

Finally, to emphasize the importance of including fiber bridging effects in the formulation, the same problem is solved and compared with and without the fiber bridging. Fig. 23 presents the force–displacement curve for fiber orientation of 90°. Clearly, neglecting the effects of fiber bridging could erroneously underestimate the load-carrying capacity of the specimen, indicating the fact that any accurate simulation of fracture in composites must include the bridging effects. In contrast to the load carrying capacity, neglecting the fiber bridging has a minor effect on the crack propagation path, as depicted in Fig. 24.

In order to demonstrate the mesh independency of the crack growth criterion, the crack path has been determined by

Fig. 22. Stress contours and the deformed configurations at \( \delta = 0.1 \) mm for the case of 90° fiber orientation: (a) \( \sigma_{xx} \) (b) \( \sigma_{yy} \) (c) \( \sigma_{xy} \) (d) deformed configuration with the magnification factor of 10.
simulating the cases of fiber orientations of 75° and 90° with three different meshes (Fig. 25). According to the results presented in Fig. 26, by refining the mesh, the crack propagation paths tend to converge to a final path, while the coarser mesh remains slightly deviated from the converged predicted crack path. This clearly shows that by enriching the displacement with asymptotic functions and using the interaction integral method for deriving fracture parameters, the numerical predictions remain insensitive to the size of the mesh and quickly converges by refining the mesh; a clear indication of mesh independence of the results.

7. Conclusion

A new formulation is proposed for fiber bridging and modeling mixed mode crack propagation in fiber reinforced composites within the framework of extended finite element method. The formulation is proved to be capable of efficiently simulating the increase in load bearing capacity and fracture toughness of fiber reinforced composites in mixed mode crack propagations with arbitrary crack paths, and improving the accuracy of the linear elastic based XFEM for modeling nonlinear fracture process zone, while eliminating the requirement of predetermined crack path in cohesive zone models.
Also, the modified maximum hoop stress criterion is introduced and utilized to predict the crack growth direction. Numerical examples of mode I and mixed mode problems are solved, and promising results have been obtained, confirming that this simple formulation can be readily used in accurate numerical simulations of fracture of fiber reinforced composites.

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Appendix A.

The finite element formulation of the XFEM is presented here [19]

\[ K_0^a = \int_{\Omega} (B^a)^T DB \, d\Omega \quad (r, b = u, a, b) \]  
\[ f^a_i = \int_{\Gamma} N_i f^a d\Gamma + \int_{\Gamma^a} N_i f^a d\Gamma \]  
\[ f^a_i = \int_{\Omega} N_i H f^a d\Omega + \int_{\Omega} N_i H f^a d\Omega \]  
\[ f^a_i = \int_{\Gamma} N_i F_j f^a d\Gamma + \int_{\Gamma^a} N_i F_j f^a d\Gamma \quad (\alpha = 1, 2, 3, \text{ and } 4) \]

In which \( f^a \) is the external force, \( f^b \) is the body force, \( D \) is the elasticity matrix, \( N_i \) are standard shape functions, and \( B_\alpha \) are matrices of shape functions derivatives, which in the case of XFEM are defined as

\[ B_1^a = \begin{bmatrix} N_{ix} & 0 \\ 0 & N_{iy} \end{bmatrix} \]  
\[ B_2^a = \begin{bmatrix} (N_i(H-H_i)_{x}) & 0 \\ 0 & (N_i(H-H_i)_{y}) \end{bmatrix} \]  
\[ B_3^a = \begin{bmatrix} (N_i(H-H_i)_{y}) & 0 \\ 0 & (N_i(H-H_i)_{x}) \end{bmatrix} \]  
\[ B_4^a = \begin{bmatrix} (N_i(F_a - F_b))_{x} & 0 \\ 0 & (N_i(F_a - F_b))_{y} \end{bmatrix} \quad (\alpha = 1, 2, 3, \text{ and } 4) \]

\[ B_5^a = \begin{bmatrix} B_1^a & B_2^a & B_3^a & B_4^a \end{bmatrix} \]

\[ B_6^a = \begin{bmatrix} (N_i(F_a - F_b))_{x} & 0 \\ 0 & (N_i(F_a - F_b))_{y} \end{bmatrix} \quad (\alpha = 1, 2, 3, \text{ and } 4) \]

where \( H \) and in above equation are the enrichment functions used to reproduce the displacement jump across the crack face and stress fields near the crack tip, respectively.

Appendix B.

Orthotropic crack tip fields used in the domain integral evaluation are presented in this section:

\[ u_{11}^{aux} = K_1 \sqrt{2\pi} Re \left\{ \frac{1}{\mu_1 - \mu_2} \left[ \mu_1 p_2 q_2(0) - \mu_2 p_1 q_1(0) \right] \right\} \]
\[ + K_2 \sqrt{2\pi} Re \left\{ \frac{1}{\mu_1 - \mu_2} \left[ p_2 q_2(0) - p_1 q_1(0) \right] \right\} \]

\[ u_{22}^{aux} = K_1 \sqrt{2\pi} Re \left\{ \frac{1}{\mu_1 - \mu_2} \left[ \mu_1 q_2(0) - \mu_2 q_1(0) \right] \right\} \]
\[ + K_2 \sqrt{2\pi} Re \left\{ \frac{1}{\mu_1 - \mu_2} \left[ q_2(0) - q_1(0) \right] \right\} \]

and the associated asymptotic crack-tip stress fields are defined as

\[ \sigma_{11}^{aux} = K_1 \sqrt{2\pi} Re \left\{ \frac{\mu_1^p p_2^p}{\mu_1^p - \mu_2^p} \left[ \mu_1^p g_{20}^p (0) - \mu_2^p g_{10}^p (0) \right] \right\} \]
\[ + K_2 \sqrt{2\pi} Re \left\{ \frac{1}{\mu_1^p - \mu_2^p} \left[ \mu_1^p g_{20}^p (0) - \mu_2^p g_{10}^p (0) \right] \right\} \]

\[ \sigma_{22}^{aux} = K_1 \sqrt{2\pi} Re \left\{ \frac{1}{\mu_1^p - \mu_2^p} \left[ \mu_1^p g_{20}^p (0) - \mu_2^p g_{10}^p (0) \right] \right\} \]
\[ + K_2 \sqrt{2\pi} Re \left\{ \frac{1}{\mu_1^p - \mu_2^p} \left[ \mu_1^p g_{20}^p (0) - \mu_2^p g_{10}^p (0) \right] \right\} \]
\[ \sigma_{12}^\text{tip} = \frac{K_1}{\sqrt{2\pi r}} \Re \left\{ \frac{1}{\mu_1} \left( \frac{\mu_3}{\mu_2} \frac{g_{12}^\text{tip}(\theta)}{g_{22}^\text{tip}(\theta)} - \frac{1}{g_{12}^\text{tip}(\theta)} \right) \right\} + \frac{K_II}{\sqrt{2\pi r}} \Re \left\{ \frac{1}{\mu_1} \left( \frac{\mu_3}{\mu_2} g_{12}^\text{tip}(\theta) - \frac{1}{g_{12}^\text{tip}(\theta)} \right) \right\} \]  

(A5)

with

\[ g_i(\theta) = \sqrt{\cos(\theta) + \mu_i \sin(\theta)} \quad (i = 1, 2) \]  

(A6)

\[ p_k = a_{11} h_k^2 + a_{12} - a_{20} h_k \quad (k = 1, 2) \]  

(A7)

\[ q_k = a_{12} h_k + a_{22} \frac{h_k}{\mu_k} \quad (k = 1, 2) \]  

(A8)

where \( a_k \) are the elements of the compliance tensor (Eq. (10)), \( \mu_k \) are the roots of the characteristic equation (Eq. (8)), and (\( r, \theta \)) are the polar coordinates measured from the crack tip (Fig. 1).

References