XFEM fracture analysis of orthotropic functionally graded materials

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Abstract

In the present study, the extended finite element method (XFEM) has been used for fracture analysis of orthotropic functionally graded materials. Orthotropic crack tip enrichments have been used to reproduce the singular stress field near a crack tip. Moreover, the incompatible interaction integral method has been employed to extract the stress intensity factor components. Accuracy and convergence of the proposed method have been evaluated by numerical examples and quality results have been obtained by far fewer DOFs. Also, crack propagation in isotropic and orthotropic FGMs in the presence of crack tip enrichments has been investigated and various propagation criteria have been compared, and verified, if available, by experimental and numerical data in the literature. Application of XFEM in combination of the maximum circumferential tensile stress criterion for investigation of crack propagation in orthotropic FGM problems is performed for the first time.

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1. Introduction

Composites are a combination of two or more materials to achieve a better behavior in a particular engineering application. Differences in constituent material properties lead to residual stresses which way consequently cause delamination and/or laminar microcracks [1–3]. Functionally graded materials (FGMs), has been developed on the concept of smooth/continuous variation of mechanical properties [4] to reduce the mentioned difficulties and disadvantages of ordinary composites by improving a number of desired properties (such as toughness and resilience to reduce concentration of stress in the interface zone) and to increase resistance of material against various types of mechanical, thermal and chemical failures [5]. Thermal covers, piezoelectric and thermo elastic applications, medical equipment and structures under non-uniform loading are frequently built from FGMs.

It has been proved that the order of singularity of stress field near a crack tip in non-homogeneous materials is the same as homogeneous materials [6,7]. Delale and Erdogan [6] investigated the crack problem in isotropic FGMs by exponential variation in material properties. Also, surface cracks in the graded coating bonded to a homogeneous substrate were studied in the first mode [8] and mixed mode [9]. Various methods have already been used to compute the stress intensity factors in FGMs. Gu et al. [10] calculated the crack tip field by a domain integral method and Anlas et al. [11] considered the modified integral method to compute the mode I stress intensity factor. The interaction integral method has been frequently used to extract components of various SIF modes in isotropic and orthotropic homogeneous materials [12]. It was also extended to bimaterial interface cracks in 3D problems [13,14]. Dolbow and Goss [15] extended the interaction integral method to extract modes I and II in isotropic FGM materials. They used asymptotic stress and displacement fields near a crack tip in an isotropic homogeneous material as an auxiliary field in the isotropic FGM problem. The incompatibilities of FGM problem and homogeneous auxiliary fields appear in the added incompatible terms in the interaction integral. Moreover, Kim and Paulino [16] investigated the path independent \( J_k \)-integral, the modified crack closure method (MCC) and the displacement correlation technique. Also, a number of recent studies have focused on 3D modeling of isotropic FGMs [17–19].

Ozturk and Erdogan examined the early development of analyt- ical orthotropic FGM in mode I and mixed mode [20,21] and Gu and Asaro [22] studied the four point bending specimen problem. Kim and Paulino [23] implemented the interaction integral method for mixed mode orthotropic FGM problems in the FEM framework. In addition, they investigated the path independent \( J_k \)-integral [24] and the modified crack closure method [25] for orthotropic FGMs. Also, Dag et al. [26] proposed a partition of unity finite element approach for mixed mode analysis of FGMs.

The extended finite element method (XFEM) is a powerful numerical approach that has been successfully used to model discontinuities. Its basic concept is to enrich the local solution by applying the Partition of Unity (PU) framework to the standard finite element method. For example, in crack modeling, by employing the discontinuous heaviside function, the displacement
discontinuity around a crack can be modeled without considering the crack surfaces, as a geometric boundary, to match element edges. Also, the singular stress field near a crack tip can be reproduced by applying the asymptotic displacement functions. After successful application of original idea of XFEM for modeling 2D problems [27–29], it was soon extended to general 3D [30–36], plate [37] and shell [38–40] problems.

Also, this method was used by Dolbow and Gosz [15] to model crack in an isotropic FGM plate analysis [41]. In order to take account the effect of orthotropy, orthotropic crack tip enrichments were developed for XFEM and successfully used in static [42–44] and dynamic crack propagation problems [45,46] of orthotropic media. Recently, the automatic enrichment technique has also been developed to numerically determine the required enrichment functions for each problem [47–51]. Moreover, new crack tip enrichments have been presented by Esna Ashari and Mohammadi [52] for interlaminar cracks in orthotropic bimaterial domains. To the best knowledge of the authors, the few available FGM related XFEM studies are limited to isotropic case and no investigation is available for modeling orthotropic FGMs by XFEM. This is the main focus of the present study on employing orthotropic XFEM for modeling crack in orthotropic FGM problems. Here, orthotropic crack tip enrichments are used to accurately estimate the crack tip variable field. In addition, the interaction integral method is adopted to extract the stress intensity factors in mixed mode problems. Also, an appropriate crack propagation criterion for orthotropic media is used to study the crack propagation phenomena in orthotropic FGMs.

In this paper, first the basic governing equations are briefly explained and the stress/displacement fields, auxiliary fields for computing the J integral and the enrichment functions are presented. Then, the basics of XFEM, blending elements and crack tip enrichments are discussed in Section 3. Also, computation of the stress intensity factors is explained in Section 4. For the sake of simplicity, Sections 1–4 are expressed for a homogeneous medium and the necessary changes for orthotropic FGMs are then presented in Section 5. Numerical integration and crack propagation criteria are discussed in Sections 6 and 7, respectively. Eventually, various numerical examples are investigated in Section 8, followed by the concluding remarks.

2. Governing equations

The Lekhnitskii form of Hook’s law for plane stress–strain relationship is written as [53]:

\[ \varepsilon_{\alpha \beta} = a_{\alpha \beta} \sigma_{\alpha \beta} \quad (\alpha, \beta = 1, 2, 6) \]  

where

\[ \varepsilon_1 = \varepsilon_{11}, \quad \varepsilon_2 = \varepsilon_{22}, \quad \varepsilon_6 = 2 \varepsilon_{12} \]  

\[ \sigma_1 = \sigma_{11}, \quad \sigma_2 = \sigma_{22}, \quad \sigma_6 = \sigma_{12} \]  

For a plane strain case, \( a_{\alpha \beta} \) are replaced by,

\[ \left( a_{\alpha \beta} - \frac{a_{\alpha 6} a_{6 \beta}}{a_{66}} \right) a_{ij} \]  

Coefficients \( a_{\alpha \beta} \) are related to components of the material compliance tensor \( s_{\alpha \beta \gamma} \),

\[ \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix} = \begin{bmatrix} 5s_{1111} & 5s_{1122} & 2s_{1112} \\ 5s_{2211} & 5s_{2222} & 2s_{2212} \\ 2s_{1211} & 2s_{1222} & 4s_{1212} \end{bmatrix} \]  

where

\[ \varepsilon_{ij} = s_{\alpha \beta \gamma} \sigma_{\alpha \beta} \quad (i, j, k, l = 1, 2, 3) \]

Using the basic theories of elasticity and applying the stress function \( \phi = \phi(x + jy) \) in an anisotropic case, the following characteristic equation is obtained

\[ a_{11} \mu^4 - 2a_{16} \mu^3 + (2a_{12} + a_{66}) \mu^2 - 2a_{26} \mu + a_{22} = 0 \]  

It can be shown that the roots of this equation are in the complex form [53]. These roots are always conjugate pairs, \( \mu_1, \mu_2, \mu_3, \mu_4 \), with the general forms of either

\[ \mu_1 = \alpha + j \beta \]  

or

\[ \mu_2 = \alpha - j \beta \]

For an orthotropic case in the plane problems, the characteristic equation is reduced to the following simplified equation [54]:

\[ a_{11} \mu^4 + (2a_{12} + a_{66}) \mu^2 + a_{22} = 0 \]  

Roots \( \mu_1 \) of this equation are purely imaginary, and can be written as \( \mu_1 = \sqrt{j}, j = 1, 2, 3, 4 \)

\[ \gamma_1^2 = -\frac{1}{2a_{12}} \left\{ -(2a_{12} + a_{66}) \pm \sqrt{(2a_{12} + a_{66})^2 - 4a_{12}a_{22}} \right\} \]

For an isotropic case, these roots reduce to \( \gamma_1 = \gamma_2 = 1 \).

2.1. Stress and displacement field

Considering (X, Y) as the global coordinate system and (x, y) as the local crack tip system, as depicted in Fig. 1, which defines the local crack tip coordinate system (r, \( \theta \)) by \( x + iy = r e^{i\theta} \), the asymptotic stress and displacement crack tip field have been derived by Sih et al. [55]. The displacement fields are,

\[ u_1 = K_1 \sqrt{2\pi} \frac{r}{\mu_1} \left\{ \frac{1}{\mu_1 - \mu_2} |\mu_1 p_1 \delta(\theta) - \mu_2 p_1 \delta(\theta)\} \right\} + K_B \]

\[ \times \sqrt{2\pi} \frac{r}{\mu_1 - \mu_2} \left\{ \frac{1}{\mu_1 - \mu_2} |p_2 \delta(\theta) - p_1 \delta(\theta)| \right\} \]  

\[ u_2 = K_1 \sqrt{2\pi} \frac{r}{\mu_1} \left\{ \frac{1}{\mu_1 - \mu_2} |\mu_1 q_1 \delta(\theta) - \mu_2 q_1 \delta(\theta)| \right\} + K_B \]

\[ \times \sqrt{2\pi} \frac{r}{\mu_1 - \mu_2} \left\{ \frac{1}{\mu_1 - \mu_2} |q_2 \delta(\theta) - q_1 \delta(\theta)| \right\} \]  

Fig. 1. Global and local crack tip coordinate systems in Cartesian and polar forms for an arbitrary orthotropic body.
where \( u_1 \) and \( u_2 \) are the displacement components in \( x \) and \( y \) directions, respectively. \( Re \) denotes the real part of complex functions and

\[
g_i(\theta) = \sqrt{\cos(\theta) + \mu_i \sin(\theta)} \quad (i = 1, 2)
\]

(14)

\[
p_k = a_{11} \mu_k^2 + a_{12} - a_{16} \mu_k \quad (k = 1, 2)
\]

(15)

\[
q_k = a_{12} \mu_k + a_{22} - a_{26} \quad (k = 1, 2)
\]

(16)

Also, the asymptotic stress components are written as

\[
\sigma_{11} = \frac{K_t}{2\pi r} \left\{ \frac{\mu_1 \mu_2}{\mu_1 - \mu_2} \left[ \frac{1}{g_2(\theta)} - \frac{1}{g_1(\theta)} \right] \right\}
\]

(17)

\[
\sigma_{22} = \frac{K_t}{2\pi r} \left\{ \frac{1}{\mu_1 - \mu_2} \left[ \frac{\mu_1}{g_2(\theta)} - \frac{1}{g_1(\theta)} \right] \right\}
\]

(18)

\[
\sigma_{12} = \frac{K_t}{2\pi r} \left\{ \frac{1}{\mu_1 - \mu_2} \left[ \frac{1}{g_2(\theta)} - \frac{\mu_1}{g_1(\theta)} \right] \right\}
\]

(19)

It should be noted that in FGM materials \( a_k \) are different from one point to another, which leads to different values of \( p_k, q_k \), and \( \mu_k \) at different points. For this reason, material properties for the auxiliary field (in contour integral) and crack tip enrichment functions are calculated at the crack tip. Consequently, the following replacement is adopted in Eqs. (12)–(19) for FGM materials

\[
\chi_k \rightarrow \chi_k^{up}
\]

(20)

where \( \chi_k \) denotes the \( a_k, p_k, q_k \) and \( \mu_k \), and \( \chi_k^{up} \) denotes the same parameters at the crack tip.

3. Extended finite element method

The extended finite element method (XFEM) somehow uses of the concepts of meshless methods within the standard finite element to efficiently model crack problems that involve displacement discontinuity and singular stress field. Reproduction of desired stress and displacement fields in XFEM is obtained by the concept of Partition of Unity (PU) [56,57]. According to the PU property, any function \( g_k \), which satisfies

\[
\sum_{k=1}^{n} g_k(x) = 1 \quad (x \in \Omega_{pu})
\]

(21)
can be easily proved to possess the reproducing property for an arbitrary function \( \psi \) in domain \( \Omega_{pu} \)

\[
\sum_{k=1}^{n} g_k(x)\psi(x) = \psi(x) \quad (x \in \Omega_{pu})
\]

(22)

Since the set of FEM shape functions satisfy Eq. (21), \( \psi \) can be used as a local enrichment basis function for variable fields in the domain \( \Omega_{enr} \)

\[
\phi^{enr} = \sum_{i=N_{enr}} N_i(x)\psi(x)q_i \quad (x \in \Omega_{enr})
\]

(23)

where \( N_{enr} \) is the set of enriched nodes, \( N_i \) is the \( i \)th shape function and \( q_i \) are additional DOFs. Assuming the enrichment basis functions \( \psi_m \) belong to the set \( M \)

\[
M = \{ \psi_1, \psi_2, \ldots, \psi_m \}
\]

(24)

Eq. (23) is modified to

\[
\phi^{enr} = \sum_{i=N_{enr}} N_i(x) \left( \sum_{m=M} \psi_m(x)q_m \right) \quad (x \in \Omega_{enr})
\]

(25)

XFEM can be used to simulate a discontinuous field, avoiding remeshing in crack propagation problems, and/or reproducing singular stress field at a crack tip using high order enrichment functions. To overcome the incompatibility between enriched and non-enriched domains, a transition zone is defined between them. Consequently, four types of subdomains are considered to model a general crack problem: the standard FEM domain, elements cut by a crack and enriched by the Heaviside function, crack tip enrichment domain and the transition (blending) region, as depicted in Fig. 2. The displacement field in XFEM can then be written as

\[
u = \mathbf{u}^{FEM} + \mathbf{u}^{XFEM}
\]

(26)

with

\[\text{Fig. 2. Definition of various elements in XFEM modeling of crack.}\]
\[ \mathbf{u}^{\text{XFEM}} = \mathbf{u}^{\text{up}} + \mathbf{u}^{\text{He}} + \mathbf{u}^{\text{Blend}} \]  
(27)

where \( \mathbf{u}^{\text{up}}, \mathbf{u}^{\text{He}} \) and \( \mathbf{u}^{\text{Blend}} \) are defined in the following sections. It should be noted that the geometry of domain in XFEM is discretized similar to conventional finite element method,

\[ \mathbf{x} = \sum_{i \in \mathcal{N}} N_i(\xi, \eta) \mathbf{x}_i \]  
(28)

### 3.1. Standard finite element method

In the isoparametric finite element method, the displacement field \( \mathbf{u}^T = (u_x, u_y) \) in an element is obtained from

\[ \mathbf{u}^T = \sum_{i \in \mathcal{N}} N_i(\xi, \eta) \mathbf{u}_i \]  
(29)

where \( \mathbf{u}_i \) is the displacement vector at node \( i \).

### 3.2. Strong discontinuity enrichment

The heaviside function is used as the enrichment for modeling the strong discontinuity in XFEM,

\[ H(\xi) = \begin{cases} 1 & \forall \xi > 0 \\ -1 & \forall \xi < 0 \end{cases} \]  
(30)

where the sign distance function is used to compute \( \zeta(\mathbf{x}) \), as depicted in Fig. 3. For a point \( \mathbf{x} \) in the Heaviside enriched domain, and \( \mathbf{x}_f \) as the projection of point \( \mathbf{x} \) on the crack, \( \zeta(\mathbf{x}) \) is defined as

\[ \zeta(\mathbf{x}) = \mathbf{d} \cdot \mathbf{n}_f \]  
(31)

where

\[ \mathbf{d} = \mathbf{x} - \mathbf{x}_f \]  
(32)

and the unit normal vector of crack line at \( \mathbf{x}_f \) is denoted by \( \mathbf{n}_f \). If the set of nodes enriched by the heaviside functions are represented by \( \mathcal{N} \), \( \mathbf{u}^{\text{He}} \) is then approximated by

\[ \mathbf{u}^{\text{He}} = \sum_{i \in \mathcal{N}} N_i(\mathbf{x}) H(\zeta) \mathbf{u}_i \]  
(33)

### 3.3. Crack tip enrichments

The nature of stress and displacement fields near the crack tip is strongly nonlinear and therefore the standard polynomial shape functions are unable to approximate them accurately. These fields can be estimated with higher accuracy if the enrichment approximation is similar to the analytical nature of the fields. If the set of crack tip enrichments is defined by

\[ F = \{ f_1, f_2, \ldots, f_m \} \]  
(34)

the crack tip displacement field is approximated by

\[ \mathbf{u}^{\text{up}} = \sum_{i \in \mathcal{T}} \mathbf{N}_i(\mathbf{x}) \left( \sum_{j=1}^{k} f_j(\mathbf{x}) \mathbf{b}_j \right) \]  
(35)

where \( \mathcal{T} \) is the set of nodes enriched by crack tip enrichment functions and \( \mathbf{b}_j \) are the added virtual DOFs. For a crack tip in an isotropic homogeneous material, the tip enrichment functions are

\[ F = \left\{ \sqrt{r} \sin(\frac{\theta}{2}) \sqrt{r} \cos(\frac{\theta}{2}) \sqrt{r} \sin(\frac{\theta}{2}) \sin(\theta) \right\} \]  
(36)

Selection of crack tip enrichment in orthotropic FGM is explained in Section 3.6.

### 3.4. Transition domain

Many researchers have studied the transition between the enriched and non-enriched domains. Here, the method presented in [58] has been employed. According to this approach, virtual hierarchical nodes are added to edges that connect the tip enrichment nodes to other cases. Consequently, two types of hierarchical nodes exist: Hierarchical nodes that correspond to edges connecting non-enriched and tip enriched nodes (defined by \( \mathcal{B}_n \)), and those edges that connect tip and Heaviside enriched nodes (defined by \( \mathcal{B}_h \)). The displacement field associated with the added hierarchical nodes can be approximated by [58]:

\[ \mathbf{u}^{\text{Blend}} = \sum_{i \in \mathcal{B}_n} \tilde{N}_i(\mathbf{x}) \mathbf{d}_i + \sum_{j \in \mathcal{B}_h} \tilde{N}_j(\mathbf{x}) H(\mathbf{x}) \mathbf{e}_j \]  
(37)

where \( \mathbf{d}_i \) and \( \mathbf{e}_j \) are added DOFs which correspond to hierarchical nodes in the sets \( \mathcal{B}_n \) and \( \mathcal{B}_h \), respectively. Also, \( \mathcal{N} \) are the hierarchical shape functions explained in [58]. For example, for nodes \( p_1 \) and \( p_2 \) in Fig. 4, the hierarchical shape functions are defined as

\[ \tilde{N}_{11}(\xi, \eta) = \frac{1 - \eta}{2} \phi_2 \xi \]  
(38)

\[ \tilde{N}_{12}(\xi, \eta) = \frac{1 - \xi}{2} \phi_2 \eta \]

where \( \xi \) and \( \eta \) are the isoparametric coordinates and \( \phi_2(\xi) \) is a polynomial of degree \( j \), usually defined in terms of the Legendre polynomial. For \( j = 2 \),

\[ \phi_2(\xi) = \frac{1}{2} \sqrt{\frac{3}{2}} (1 - \xi^2) \]  
(39)

Further details of these functions and triangular elements can be found in [58,59].

### 3.5. XFEM displacement field

Combining Eqs. (26), (27), (29), (33), (35) and (37), the displacement field can be generally approximated by:

\[ \mathbf{u}(\mathbf{x}) = \left[ \sum_{i \in \mathcal{N}} \mathbf{N}_i(\mathbf{x}) \mathbf{u}_i \right] + \left[ \sum_{i \in \mathcal{N}} \mathbf{N}_i(\mathbf{x}) H(\zeta(\mathbf{x})) \mathbf{d}_i \right] \]

\[ + \sum_{j \in \mathcal{T}} \tilde{N}_j(\mathbf{x}) \mathbf{d}_j + \sum_{k \in \mathcal{B}_h} \tilde{N}_k(\mathbf{x}) H(\mathbf{x}) \mathbf{e}_k \]  
(40)

where the four brackets [ ] represent the linear, discontinuous, tip enrichment, and transition parts of approximation, respectively.
3.6. Orthotropic enrichment functions

Asadpoure et al. [42–44] investigated various types of crack tip enrichment functions in orthotropic media. In the most general case, presented in [44], the orthotropic crack tip enrichment functions in the crack tip local polar coordinate system \((r, \theta)\) are defined as

\[
F(r, \theta) = \left\{ \begin{array}{l}
\sqrt{r} \cos \left( \frac{\theta}{2} \right) \sqrt{g_1(\theta)}, \quad \sqrt{r} \sin \left( \frac{\theta}{2} \right) \sqrt{g_2(\theta)}, \quad \sqrt{r} \sin \left( \frac{\theta}{2} \right) \sqrt{g_3(\theta)}
\end{array} \right.
\]

where

\[
g_j(\theta) = \left( \cos(\theta) + \beta_j \sin(\theta) \right)^2 + \left( \beta_j \sin(\theta) \right)^2 \quad (j = 1, 2)
\]

and \(\beta_1\) and \(\beta_2\) have been defined in (9). According to discussions of Dolbow and Gosz [15] for isotropic FGMs and Kim and Paulino [23] for orthotropic FGMs, the asymptotic crack tip fields can be used as the auxiliary fields for computation of SIFs in orthotropic FGM problems.

4. Stress intensity factor

4.1. J integral

The \(J\) integral has been used in this study to compute the stress intensity factor. Three different formulations have been proposed in the literature: non-equilibrium, incompatibility, and constant-constitutive-tensor [60]. The first one only satisfies the constitutive and compatibility equations

\[
\sigma_{ij} = c_{ijkl}(X)\varepsilon_{kl}, \quad \varepsilon_{ij} = \frac{1}{2} \left( u_{ij} + u_{ji} \right), \quad \sigma_{ij} \neq 0
\]

The second approach satisfies only equilibrium and the constitutive equations

\[
\sigma_{ij} = c_{ijkl}(X)\varepsilon_{kl}, \quad \varepsilon_{ij} = \frac{1}{2} \left( u_{ij} + u_{ji} \right), \quad \sigma_{ij} = 0
\]

and the constant-constitutive-tensor only satisfies equilibrium and compatibility

\[
\sigma_{ij} = c_{ijkl}(X)\varepsilon_{kl}, \quad \varepsilon_{ij} = \frac{1}{2} \left( u_{ij} + u_{ji} \right), \quad \sigma_{ij} = 0
\]

where \(c_{ijkl}\) is the material modulus. For detailed formulation of each case refer to [60]. While the constant-constitutive-tensor formulation needs derivatives of stress and strain in actual fields which leads to inaccuracies in \(C^0\) finite element formulation, the two other cases have the same accuracy [60]; hence the incompatibility formulation, which requires less complicated derivatives, is considered here. The strain tensor is calculated from

\[
\varepsilon_{ij} = \delta_{ij}(X)\sigma_{ij}
\]

where \(\mathbf{s} = \mathbf{c}^{-1}\). The general form of path independent \(J\) integral is written as [61]

\[
J = \int_{\Gamma} \left( 1 - w \frac{\partial \varepsilon_{ij}}{\partial x_i} \right) n_j d\Gamma
\]

where \(\Gamma\) is an arbitrary contour surrounding the crack tip, \(n_j\) is the \(j\)th component of the outward unit normal to \(\Gamma\). \(\delta_j\) is the Kronecker delta and the Cartesian coordinate system whose \(x\) axis is parallel to the crack surface is considered. \(w\) is the strain energy density,

\[
w = \frac{1}{2} c_{ijkl} \varepsilon_{ij} \varepsilon_{kl}
\]

Using the equivalent domain integral (Fig. 5), Eq. (48) is transformed to

\[
J = \int_A \left( \sigma_{ij} u_{ij} - w \delta_{ij} \right) q dA + \int_A \left( \sigma_{ij} u_{ij} - w \delta_{ij} \right) q dA
\]

where \(q\) is a smooth function from \(q = 1\) on interior boundary of \(A\) and \(q = 0\) on the outer one, as depicted in Fig. 5.

4.2. Separation of stress intensity factors

The interaction integral is employed to compute modes I and II stress intensity factors. It is based on superimposition of auxiliary and actual fields,

\[
f^* = f + f_{\text{aux}} + M^f
\]

where \(f\) and \(f_{\text{aux}}\) are the \(f\)-integrals corresponding to actual and auxiliary fields, respectively. \(f_{\text{aux}}\) is defined by

\[
f_{\text{aux}} = \int_A \left( \sigma_{ij}^{\text{aux}} u_{ij}^{\text{aux}} - w^{\text{aux}} \delta_{ij} \right) q dA
\]

and \(M^f\) is the local interaction integral calculated by

\[
M^f = \int_A \left( \sigma_{ij}^{\text{aux}} u_{ij}^{\text{aux}} - w^{\text{aux}} \delta_{ij} \right) q dA
\]
\[ M' = \int_{A} \left\{ \sigma_{i k} u_{i 1}^{\text{aux}} + \sigma_{i k}^{\text{aux}} u_{i 1} - \frac{1}{2} (\sigma_{i k} u_{i k}^{\text{aux}} + \sigma_{i k}^{\text{aux}} u_{k i}) \right\} q_{j} dA + \int_{A} \left\{ \sigma_{i k} u_{i 1}^{\text{aux}} + \sigma_{i k}^{\text{aux}} u_{i 1} - \frac{1}{2} (\sigma_{i k} u_{i k}^{\text{aux}} + \sigma_{i k}^{\text{aux}} u_{k i}) \right\} q_{j} dA \]

Considering
\[ \sigma_{i k}^{\text{aux}} u_{i 1} = \sigma_{i k}^{\text{aux}}_{\text{eq} 1}, \quad \sigma_{i k}^{\text{aux}} = \sigma_{i k}^{\text{aux}}_{\text{eq} 0}, \quad \sigma_{i k,1}^{\text{aux}} e_{i k} = \sigma_{i k,1}^{\text{aux}} e_{i k} \]

the global interaction integral \( M' \) will be now
\[ M' = \int_{A} \left\{ \sigma_{i k} u_{i 1}^{\text{aux}} + \sigma_{i k}^{\text{aux}} u_{i 1} - \frac{1}{2} (\sigma_{i k} u_{i k}^{\text{aux}} + \sigma_{i k}^{\text{aux}} u_{k i}) \right\} q_{j} dA \]

The local \( M' \) is calculated by the following transformation
\[ M' = M_{1}^{c} \cos(\theta) + M_{2}^{c} \sin(\theta) \]

The energy release rate in elastic media is calculated as
\[ G = J = c_{11} K_{1}^{2} + c_{12} K_{1} K_{2} + c_{22} K_{2}^{2} \]  

where
\[ c_{11} = -\frac{\alpha_{22}}{2} \ln \left( \frac{\mu_{1} + \mu_{2}}{\mu_{1} \mu_{2}} \right) \]
\[ c_{12} = -\frac{\alpha_{22}}{2} \ln \left( \frac{1}{\mu_{1} \mu_{2}} \right) + \frac{\alpha_{11}}{2} \ln(\mu_{1} \mu_{2}) \]
\[ c_{22} = \frac{\alpha_{11}}{2} \ln(\mu_{1} \mu_{2}) \]

It has been proved that for the two superimposed fields \[ f' = c_{11}(K_{1}^{\text{aux}} + K_{1}) + c_{12}(K_{1}^{\text{aux}} + K_{1})(K_{2}^{\text{aux}} + K_{2}) + c_{22}(K_{2}^{\text{aux}} + K_{2}) \]
\[ J = \int \left( J^{\text{aux}} + M' \right) \]

where
\[ J^{\text{aux}} = c_{11}(K_{1}^{\text{aux}})^{2} + c_{12}(K_{1}^{\text{aux}})(K_{2}^{\text{aux}}) + c_{22}(K_{2}^{\text{aux}})^{2} \]
\[ M' = 2c_{11} K_{1}^{\text{aux}} + c_{12}(K_{1}^{\text{aux}})(K_{2}^{\text{aux}}) + 2c_{22} K_{2}^{\text{aux}} \]

Setting \( K_{1}^{\text{aux}} = 1, K_{2}^{\text{aux}} = 0 \) and \( K_{1}^{\text{aux}} = 0, K_{2}^{\text{aux}} = 1 \), leads to the following linear algebraic equations to calculate actual modes \( l \) and \( II \) stress intensity factors,
\[ \begin{cases} M_{1}^{l} = 2c_{11} K_{1} + c_{12} K_{2} & (K_{1}^{\text{aux}} = 1 \text{ and } K_{2}^{\text{aux}} = 0) \\ M_{2}^{l} = c_{12} K_{1} + 2c_{22} K_{2} & (K_{1}^{\text{aux}} = 0 \text{ and } K_{2}^{\text{aux}} = 1) \end{cases} \]

Evaluation of \( \sigma_{i k}^{\text{aux}} \) in \( M' \) (Eq. (53)) can be simplified by the method proposed in [63].
\[ \sigma_{i k}^{\text{aux}} = \sigma_{i k}^{\text{aux}} \left[ \left( c_{i k,1}^{\text{aux}} \right)_{\text{eq} 1} + \left( c_{i k,1}^{\text{aux}} \right)_{\text{eq} 0} \right] \]

\[ = \sigma_{i k}^{\text{aux}} \left[ \left( c_{i k,1}^{\text{aux}} \right)_{\text{eq} 1} + \left( c_{i k,1}^{\text{aux}} \right)_{\text{eq} 0} \right] \]

5. FGM considerations

In FGM problems, any material properties \( P \), such as modules of elasticity \( E_{11}, E_{22}, \) shear modules \( G_{12} \) and Poisson’s ratio \( \nu_{12}, \nu_{21} \) vary at different points of the domain,
\[ P(x, y) = \varphi(x, y) \]

where \( \varphi(x, y) \) is a predefined function, usually exponential or linear. Such variable properties lead to variation of constitutive tensor in different points. While in the standard FGM modeling, the material properties (68) are defined on each Gauss point, in the isoparametric graded finite element method [64], material properties are interpolated from nodal values by FEM shape functions,
\[ P(x, y) = \sum_{i=1}^{n} N_{i}(x, y) \hat{P}_{i} \]

where \( \hat{P} \) is the nodal value of property \( P \). In the interaction integral (55), \( \sigma_{i k,l} \) or \( \sigma_{i k,II} \) can be considered as a \( P \) and their derivatives can be calculated accordingly,
\[ \frac{\partial P(x, y)}{\partial x} = \sum_{i=1}^{n} \frac{\partial N_{i}(x, y)}{\partial x} \hat{P}_{i} \]

Also, the roots of Eq. (7), to be used in auxiliary fields, should be calculated at the crack tip, as described in relation (20). Values of \( \zeta_{i} \) and \( \beta_{i} \) in Eqs. (42) and (43) can also be calculated at the crack tip, but if a large area is to be enriched, \( \zeta_{i} \) and \( \beta_{i} \) should be calculated at enriched nodes and interpolated by FEM shape functions (69) and...
In an orthotropic medium, the roots of characteristic equation are obtained in closed form (11), where \( \alpha_i = 0 \) and derivatives of \( \beta \) can be directly calculated from Eq. (11).

### 6. Numerical integration

The Gauss quadrature rule has been employed for numerical integration. Four Gauss points are used in standard four-node elements and one Gauss point for three-node elements. The seventh order Gauss integration is used for non-cracked enriched elements. The sub-triangulation method is used to integrate cracked elements, in such a way that none of sub-triangles include the crack, as depicted in Fig. 6 for three and four node elements. Then, 7 Gauss points are used for each sub-triangle.

### 7. Crack propagation

#### 7.1. Isotropic criteria

In isotropic materials, various crack propagation criteria are available in the literature such as maximum hoop stress [65], maximum strain energy release rate [66] and minimum strain energy density [67]. Here, maximum hoop stress and maximum energy release rate criteria are used.

**7.1.1. Maximum hoop stress criterion**

The maximum hoop stress (maximum circumferential tensile stress) criterion is based on the assumption that the crack propagates in radial direction, perpendicular to the maximum hoop stress (\( \sigma_{max} \)) direction. The crack propagation angle \( \theta_0 \) with respect to local crack tip coordinate is obtained from

\[
\frac{K_I}{K_{I0}} \cos^3(\theta_0/2) - \frac{3}{2} \frac{K_{II}}{K_{I0}} \cos(\theta_0/2) \sin(\theta_0) = 1
\]

(71)

**7.1.2. Maximum energy release rate criterion**

Hussain et al. [66] expressed SIF values in terms of crack propagation angle \( \theta \) as

\[
K_I(\theta) = g(\theta) \left( \frac{1}{2} K_{I0} \cos(\theta) + \frac{3}{2} K_{II} \sin(\theta) \right)
\]

(72)

\[
K_I(\theta) = g(\theta) \left( K_{I0} \cos(\theta) - \frac{3}{2} K_{II} \sin(\theta) \right)
\]

(73)

where

\[
g(\theta) = \frac{4}{3 + \cos^2(\theta)} \left( 1 - \frac{\theta}{\pi} \right)^{\frac{1}{2}}
\]

(74)

Knowing

\[
G(\theta) = \frac{K_I^2 + K_{II}^2}{E_{tip}}
\]

(75)

results in

\[
G(\theta) = \frac{1}{4E_{tip}} b^2(\theta) \left( 1 + 3 \cos^2(\theta) \right) K_{I0}^2 + 8 \sin(\theta) \cos(\theta) K_{I0} K_{II}
\]

(76)

\[
+ \left( 9 - 5 \cos^2(\theta) \right) K_{II}^2
\]

where \( E_{tip} = E_{tip} \) for the plane stress and \( E_{tip} = E_{tip} / (1 - \nu_{tip}) \) for plane strain problems. Crack propagation angle is then obtained by minimizing

\[
\frac{\partial G(\theta)}{\partial \theta} = 0, \quad \frac{\partial^2 G(\theta)}{\partial \theta^2} < 0, \quad G(\theta) = G_{cr}(\theta)
\]

(77)

where \( G_{cr}(\theta) \) is the critical energy release rate

\[
G_{cr}(\theta) = \frac{K_{I0}^2}{E_{tip}}
\]

(78)

#### 7.2. Orthotropic criterion

Various orthotropic mixed mode crack propagation criteria are presented by Saouma et al. [68]. According to this method, the first mode of critical stress intensity factor in each direction is defined as

\[
K_{Icr}^0 = K_{Icr}^0 \cos^2(\beta) + K_{IIcr}^0 \sin^2(\beta)
\]

(79)

where \( K_{Icr}^0 \) and \( K_{IIcr}^0 \) are the critical stress intensity factors of mode I along \( x \) and \( y \) directions respectively. Also, \( (\beta = \theta + \epsilon_0) \) where \( \theta \) and \( \epsilon_0 \)

---

**Fig. 6.** Configuration of sub-triangulation for numerical integration in 4-node and 3-node enriched elements.

**Fig. 7.** Plate with a center crack parallel to material gradient under constant strain loading.
have been defined in Fig. 1. The direction of crack propagation is calculated by maximizing the following relation
\[ P = \operatorname{Re} \left\{ A(\mu_1 B_2 - \mu_2 B_1) \right\} + m \operatorname{Re} \left\{ A(B_2 - B_1) \right\} \]
where \( m = K_a/K_1, n = K_{aV}/K_{1V} = E_1/E_2 \) and
\[ A = \frac{1}{\mu_1 - \mu_2}, \quad B_1 = (\mu_1 \sin(\theta) + \cos(\theta))^{1.5}, \quad C = \cos^2(\theta + \omega) + n \sin^2(\theta + \omega) \]
Finally, the crack starts to propagate whenever the following equation is satisfied
\[ \frac{\sigma_\max}{\sigma_\max} = \frac{K_a \text{Re} \left\{ A(\mu_1 B_2 - \mu_2 B_1) \right\} + K_a \text{Re} \left\{ A(B_2 - B_1) \right\}}{K_{1V} \cos^2(\theta + \omega) + K_{1V} \sin^2(\theta + \omega)} = 1 \]
8. Numerical examples
8.1. Plate with a center crack parallel to material gradient
A square plate containing a center crack is considered, as depicted in Fig. 7. The crack is parallel to material gradient and a constant strain loading which corresponds to uncracked plate is applied to the top edge of plate. The boundary conditions are shown in Fig. 7. In order to apply the constant strain, a normal stress equal to \( \sigma_{22}(x, 10) = E \sigma_0(x, 10) \) is applied on the top edge. Here, the effective Young’s modules \( E \), the effective Poisson’s ratio \( v \), the stiffness ratio \( \delta \) and shear parameter \( K_0 \) are defined as
\[ E = \sqrt{E_{11}E_{22}}, \quad \sigma = \sqrt{\sigma_{12} \sigma_{21}}, \quad \delta^4 = \frac{E_{11}}{E_{22}} = \frac{v_{12}}{v_{21}}, \quad K_0 = \frac{E}{2G_{12}}, \quad \sigma_\max = \frac{\sigma_\max}{E} \]
A mesh of 1369 four-node elements and 1444 nodes is used, as depicted in Fig. 8. The exponential material properties along the x direction is assumed as
\[ E_{11}(x) = E_{11}^0 e^{\alpha x}, \quad E_{22}(x) = E_{22}^0 e^{\alpha x}, \quad G_{12}(x) = G_{12}^0 e^{\alpha x} \]
and geometrical properties are
\[ a/W = 1, \quad L/W = 1 \]

![Adaptively structured mesh of 1369 4-node elements and 1444 nodes.](image)

Table 1
The effect of material gradation on normalized stress intensity factor under constant strain loading.

<table>
<thead>
<tr>
<th>( \beta a )</th>
<th>( K_a/K_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.3996</td>
</tr>
<tr>
<td>0.2</td>
<td>1.4398</td>
</tr>
<tr>
<td>0.3</td>
<td>1.4381</td>
</tr>
<tr>
<td>0.4</td>
<td>1.4475</td>
</tr>
<tr>
<td>0.5</td>
<td>1.4435</td>
</tr>
<tr>
<td>0.7</td>
<td>1.4581</td>
</tr>
<tr>
<td>0.9</td>
<td>1.4568</td>
</tr>
</tbody>
</table>

\[ \beta a = (0.1, 0.2, 0.3, 0.4, 0.5, 0.7, 0.9), \quad K_0 = 0.5 \]

In order to investigate the effect of radius of contour integral on stress intensity factors, values of the normalized stress intensity factor with respect to \( \beta a = 0 \) by orthotropic crack tip enrichments. The same accuracy has been obtained by lower number of nodes and elements.

Also, the normalized stress intensity factor for various Poisson’s ratio and \( \beta a = 0.5 \) are shown in Table 2 and compared with those values presented in Refs. [23,20]. Fig. 9 depicts the variations of normalized stress intensity factor with respect to \( \beta a \), as explained in Table 1.

In order to investigate the effect of radius of contour integral on stress intensity factors, values of the normalized stress intensity factor with respect to \( \beta a = 0 \) by orthotropic crack tip enrichments. The same accuracy has been obtained by lower number of nodes and elements.

Also, the normalized stress intensity factor for various Poisson’s ratio and \( \beta a = 0.5 \) are shown in Table 2 and compared with those values presented in Refs. [23,20]. Fig. 9 depicts the variations of normalized stress intensity factor with respect to \( \beta a \), as explained in Table 1.
Convergence of error of normalized stress intensity factor for $\beta a = 0.5$ is depicted in Fig. 11 in a logarithmic scale. To this propose, constant enriched area with radius $r = 0.15a$ has been considered. According to Table 1, values of $k$ presented by Refs. [25,23] differ each other and to the knowledge of authors, there is no available exact solution for this problem in the literature. Therefore, the average value of normalized stress intensity factor presented in those references, which is almost equal to the value calculated by a fine mesh, has been considered as the exact value. The vertical axis expresses the error of $k$ by
\[
\text{error} = \frac{k - k_{\text{exact}}}{k_{\text{exact}}} \times 100
\] (90)

Clearly a good convergence rate is observed for the present method.

Also, variation of condition number of stiffness matrix, as an indirect index of numerical accuracy, versus the number of DOFs by

Fig. 9. Normalized stress intensity factor versus various material gradation.

Fig. 10. The effect of radius of contour integral on stress intensity factor.

Fig. 11. Convergence of error in normalized stress intensity factors.

Fig. 12. Condition number of global stiffness matrix.

Fig. 13. Geometry and the finite element mesh of a tensile plate with an inclined central crack.
has been depicted in Fig. 12. In this figure, the values of condition number of un-cracked plate have also been depicted by a dash line. It clearly shows that the differences between the condition numbers in the presence of crack and without it are negligible, particularly in the lower DOFs. Even in higher DOFs, the condition number of the stiffness matrix remains well within the acceptable range of numerical implementation.

8.2. Plate with an inclined center crack

In this example, a mixed mode fracture problem is investigated. Geometry of this problem and the finite element mesh are depicted in Fig. 13. Here, the plane stress plate under a constant strain which corresponds to the un-cracked plate is considered. This is achieved by applying the stress

\[ \sigma_{22}(x, 10) = \frac{1}{E(x, 10)} \]

on the top edge. A structured mesh of 961 four-node elements (1024 nodes) for the isotropic case and 1369 four-node elements and 1444 nodes (same as previous example) for the orthotropic one have been employed in this model. An exponential variation of material properties similar to previous problem is considered. The material properties for the isotropic case are:

\[ E_0 = 1, \ t(x) = \frac{3}{119} \]

and for the orthotropic case

\[ E_0 = 1, \ v(x) = 0.25, \ \gamma = 0.15 \]

has been depicted in Fig. 12. In this figure, the values of condition number of un-cracked plate have also been depicted by a dash line. It clearly shows that the differences between the condition numbers in the presence of crack and without it are negligible, particularly in the lower DOFs. Even in higher DOFs, the condition number of the stiffness matrix remains well within the acceptable range of numerical implementation.

### Table 3
Comparison of normalized stress intensity factors for isotropic case.

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>Present XFEM</th>
<th>Singular FEM Kim and Paulino [60]</th>
<th>Isotropic XFEM Dolbow and Goss [15]</th>
<th>Semi-analytical Konda and Erdogan [71]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( K_1/K_0 )</td>
<td>( K_2/K_0 )</td>
<td>( K_1/K_0 )</td>
<td>( K_2/K_0 )</td>
</tr>
<tr>
<td>0</td>
<td>1.4288</td>
<td>-0.0009</td>
<td>1.4234</td>
<td>0.0000</td>
</tr>
<tr>
<td>18</td>
<td>1.2851</td>
<td>0.3436</td>
<td>1.2835</td>
<td>0.3454</td>
</tr>
<tr>
<td>36</td>
<td>0.9232</td>
<td>0.5469</td>
<td>0.9224</td>
<td>0.5502</td>
</tr>
<tr>
<td>54</td>
<td>0.4845</td>
<td>0.5435</td>
<td>0.4880</td>
<td>0.5338</td>
</tr>
<tr>
<td>72</td>
<td>0.1359</td>
<td>0.3252</td>
<td>0.1451</td>
<td>0.3147</td>
</tr>
</tbody>
</table>

### Table 4
Comparison of normalized stress intensity factor for orthotropic case.

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>Present XFEM</th>
<th>Kim and Paulino [60]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( K_1/K_0 )</td>
<td>( K_2/K_0 )</td>
</tr>
<tr>
<td>0</td>
<td>1.429</td>
<td>0.0000</td>
</tr>
<tr>
<td>18</td>
<td>1.329</td>
<td>0.246</td>
</tr>
<tr>
<td>36</td>
<td>1.010</td>
<td>0.411</td>
</tr>
<tr>
<td>54</td>
<td>0.587</td>
<td>0.443</td>
</tr>
<tr>
<td>72</td>
<td>0.216</td>
<td>0.3050</td>
</tr>
</tbody>
</table>

### Table 5
Comparison of stress intensity factor in proportional and non-proportional variation of material properties.

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( \gamma )</th>
<th>Present XFEM</th>
<th>Ref. [24]</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( K_1 )</td>
<td>( K_2 )</td>
<td>( E_0 )</td>
</tr>
<tr>
<td>0.2</td>
<td>0.25</td>
<td>0.15</td>
<td>1.741</td>
<td>-0.951</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0.2</td>
<td>1.171</td>
<td>-0.872</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>1.330</td>
<td>-0.940</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
<td>1.497</td>
<td>-0.994</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>1.669</td>
<td>-1.040</td>
</tr>
<tr>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>1.861</td>
<td>-1.091</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2.077</td>
<td>-1.151</td>
</tr>
</tbody>
</table>
The isotropic case was previously studied by Konda and Erdogan [71] using a semi-analytical method, Dolbow and Gosz [15] by the extended finite element method and Kim and Paulino [60] by singular finite element methods a mesh of 5336. Table 3 shows the normalized stress intensity factors of the isotropic case at the right crack tip for the present method and compares them by the mentioned references which shows a very close agreement. In this table, the normalized SIF is defined similar to the previous example. Fig. 14 depicts the variation of $J$ (Eq. (57)) versus the crack angle which shows a decrease in $J$ by increasing angle ($\theta$).

Results for the orthotropic case have been shown in Table 4. A good agreement can be seen between mode I of SIFs even by fewer number of nodes but, the estimation of the present method for mode II of SIF are slightly more different than the values presented by Ref. [60].

\[
E_{11}^0 = 10^4, \quad E_{22}^0 = 10^3, \quad G_{12}^0 = 1216, \quad v_{12} = 0.3 \quad (92)
\]

Fig. 17. Geometry and boundary condition of the four point bending specimen.

![Fig. 17](image1)

Fig. 18. The finite element mesh for modeling the four point bending specimen.

![Fig. 18](image2)

Fig. 19. Normalized norm of complex mixed mode SIF for the four point bending specimen.

![Fig. 19](image3)

Fig. 20. Phase angle of complex mixed mode SIF for the four point bending specimen.

![Fig. 20](image4)

### Table 6

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$K_I$ Ref. [22]</th>
<th>Present XFEM</th>
<th>Error (%)</th>
<th>$\psi = \tan^{-1}(K_{II}/K_I)$ Ref. [22]</th>
<th>Present XFEM</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.33</td>
<td>1.34</td>
<td>0.82</td>
<td>57.19</td>
<td>55.78</td>
<td>2.46</td>
</tr>
<tr>
<td>0.4</td>
<td>1.53</td>
<td>1.52</td>
<td>0.55</td>
<td>58.02</td>
<td>60.21</td>
<td>3.78</td>
</tr>
<tr>
<td>0.8</td>
<td>1.68</td>
<td>1.64</td>
<td>2.49</td>
<td>58.71</td>
<td>58.28</td>
<td>0.73</td>
</tr>
<tr>
<td>1.2</td>
<td>1.90</td>
<td>1.92</td>
<td>1.18</td>
<td>59.39</td>
<td>59.05</td>
<td>0.57</td>
</tr>
<tr>
<td>1.6</td>
<td>2.16</td>
<td>2.12</td>
<td>1.60</td>
<td>59.94</td>
<td>59.57</td>
<td>0.63</td>
</tr>
<tr>
<td>2</td>
<td>2.42</td>
<td>2.42</td>
<td>0.17</td>
<td>60.22</td>
<td>61.24</td>
<td>1.69</td>
</tr>
<tr>
<td>2.4</td>
<td>2.68</td>
<td>2.64</td>
<td>1.71</td>
<td>60.50</td>
<td>61.49</td>
<td>1.64</td>
</tr>
</tbody>
</table>
Also, Fig. 15 shows variations of the normalized \( J \) values for various radii of contour integrals, clearly indicating the insensitivity of \( J \) for different integral domains.

In a different case, a non-proportional mixed mode crack \((\theta = -36^\circ)\) in an FGM square plate is investigated. Material functions are [24]

\[
E_{11}(x) = E_{11}^0 e^{ax}, \quad E_{22}(x) = E_{22}^0 e^{bx}, \quad G_{12}(x) = G_{12}^0 e^{cx}
\]

where

\[
E_{11}^0 = 0.75, \quad E_{22}^0 = 1, \quad G_{12}^0 = 0.5, \quad v_{12} = 0.3
\]

\[2a = 2, \quad L/W = 1, \quad a/W = 1\]

The non-proportional constant is considered as \((\alpha, \beta, \gamma) = (0, 0.25, 0.15)\) [24]. Table 5 compares the results of non-proportional and various proportional modelings. As it can be clearly observed, for the non-proportional case almost the same values have been obtained for modes \( I \) and \( II \) with acceptable level of error: 0.096% and 6.213%, respectively. It should be noted that the reference results were obtained by 2392 nodes, almost twice the present simulation.

Table 7
Norm of complex SIF for different types of crack tip enrichments \( \beta = 1 \).

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>Ref. [22]</th>
<th>Type of enrichment</th>
<th>Orthotropic enr.</th>
<th>Isotropic enr.</th>
<th>No enr.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>(</td>
<td>K</td>
<td>)</td>
<td>Error (%)</td>
</tr>
<tr>
<td>0.0</td>
<td>1.33</td>
<td>1.34</td>
<td>0.82</td>
<td>1.69</td>
<td>27.31</td>
</tr>
<tr>
<td>0.1</td>
<td>1.53</td>
<td>1.52</td>
<td>0.55</td>
<td>1.97</td>
<td>28.90</td>
</tr>
<tr>
<td>0.2</td>
<td>1.66</td>
<td>1.64</td>
<td>2.40</td>
<td>2.22</td>
<td>32.19</td>
</tr>
<tr>
<td>0.3</td>
<td>1.90</td>
<td>1.92</td>
<td>1.18</td>
<td>2.50</td>
<td>31.73</td>
</tr>
<tr>
<td>0.4</td>
<td>2.16</td>
<td>2.12</td>
<td>1.60</td>
<td>2.90</td>
<td>34.33</td>
</tr>
<tr>
<td>0.5</td>
<td>2.42</td>
<td>2.42</td>
<td>0.17</td>
<td>3.29</td>
<td>36.04</td>
</tr>
<tr>
<td>0.6</td>
<td>2.68</td>
<td>2.64</td>
<td>1.71</td>
<td>3.64</td>
<td>35.80</td>
</tr>
</tbody>
</table>

Table 8
Phase angle \( \psi = \tan^{-1}(K_0/K) \) for different types of crack tip enrichments \( \beta = 1 \).

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>Ref. [22]</th>
<th>Type of enrichment</th>
<th>Orthotropic enr.</th>
<th>Isotropic enr.</th>
<th>No enr.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( \psi )</td>
<td>Error (%)</td>
<td>( \psi )</td>
<td>Error (%)</td>
</tr>
<tr>
<td>0.0</td>
<td>57.19</td>
<td>55.78</td>
<td>2.46</td>
<td>61.44</td>
<td>7.43</td>
</tr>
<tr>
<td>0.1</td>
<td>58.02</td>
<td>60.21</td>
<td>3.78</td>
<td>65.30</td>
<td>12.55</td>
</tr>
<tr>
<td>0.2</td>
<td>58.71</td>
<td>58.28</td>
<td>0.73</td>
<td>64.96</td>
<td>10.65</td>
</tr>
<tr>
<td>0.3</td>
<td>59.39</td>
<td>59.05</td>
<td>0.57</td>
<td>59.91</td>
<td>0.87</td>
</tr>
<tr>
<td>0.4</td>
<td>59.94</td>
<td>59.57</td>
<td>0.63</td>
<td>63.56</td>
<td>6.03</td>
</tr>
<tr>
<td>0.5</td>
<td>60.22</td>
<td>61.24</td>
<td>1.69</td>
<td>64.37</td>
<td>6.90</td>
</tr>
<tr>
<td>0.6</td>
<td>60.50</td>
<td>61.49</td>
<td>1.64</td>
<td>64.51</td>
<td>6.64</td>
</tr>
</tbody>
</table>

Table 9
Stress intensity factor for mixed mode crack in a rectangular plate.

<table>
<thead>
<tr>
<th>Material properties</th>
<th>Stress intensity factor</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>( \beta )</td>
<td>( \gamma )</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>0.5</td>
<td>0.4</td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.4</td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Also, Fig. 15 shows variations of the normalized \( J \) values for various radii of contour integrals, clearly indicating the insensitivity of \( J \) for different integral domains.
Fig. 16 shows the variation of energy release rate (Eq. (57)) for different radius of contour of $J$ integral. For $r > 0.5$ contours, an almost constant $G$ is obtained.

8.3. Four-point bending specimen

Gu and Asaro [22] investigated the mixed mode stress intensity factor in a four-point FGM bending specimen, considering exponential variation of material properties: modulus of elasticity, shear modulus and Poisson’s ratio in $y$ direction as

$$ E_{11}(y) = E_{01} e^{by}, \quad E_{22}(y) = E_{02} e^{by}, \quad \lambda = \frac{E_{22}(y)}{E_{11}(y)} $$

$$ v_{12}(y) = v_{01}(1 + cy)e^{by}, \quad v_{21}(y) = v_{02}(1 + cy)e^{by} $$

$$ G_{12}(y) = \frac{E_{22}(y)}{2(\sqrt{\lambda} + v_{21}(y))} $$

$$ \nu = 0.3, \quad \chi = -0.9, \quad P = 1 $$

where $\nu$ is the effective Poisson ratio, as defined by Eq. (83). Geometry and boundary conditions of the problem are depicted in Fig. 17.

According to the discussion by Gu and Asaro [22], the complex stress intensity factor is defined as $K = K_I + iK_{II}$, and the norm of SIF and phase angle of complex stress intensity factor can be expressed as

$$ K = |K|e^{i\psi}, \quad |K| = \sqrt{K_I^2 + K_{II}^2} $$

$$ \psi = \tan^{-1} \left( \frac{K_{II}}{K_I} \right) $$

Considering the symmetry of specimen with respect to $y$ axis, only one half of the specimen has been modeled by 751 three-node elements and 422 nodes (1008 DOFs). The configuration of mesh is depicted in Fig. 18.

The calculated norm of complex stress intensity factor for $\lambda = 0.1$ has been compared with the reference results [22] in Table 6. Also, Kim and Paulino [23] modeled the whole of specimen by singular elements and 2319 nodes (4638 DOFs). They obtained similar results to Gu and Asaro [22]. It is observed that singular FEM required about 2.75 (2.3) times more nodes (DOFs) than XFEM for a more or less similar level of accuracy, which shows the
superiority of XFEM. Normalized values of norm of complex SIF are depicted in Fig. 19, which show a good agreement between the present method and references results.

Here, the effect of various types of crack tip enrichments is considered. Generally, three cases are considered: orthotropic crack tip enrichments, isotropic one and no crack tip enrichment. In the first two cases, the radius of tip enriched area remains equal to \( r = 0.1a \) \((a = 3)\). Tables 7 and 8 compare the norm of complex SIF and phase angle of SIFs for different enrichment strategies.

Also, variations of normalized values of complex SIFs and their phase angles have been depicted in Figs. 20 and 21, respectively. The maximum errors of norm of complex SIF for various types of tip enrichments are 2.49%, 36.04% and 28.5% for orthotropic enrichment, isotropic one and no crack tip enrichment, respectively, while the maximum errors of phase angle of SIFs are 2.49%, 36.04% and 28.5% for orthotropic enrichment, isotropic one and no crack tip enrichment, respectively.
the graded domain are presented in Table 11. The two edge cracks are emanated from the lower edge at $n = 0.17$ and $n = 0.58$. 4848 three-node elements and 2547 nodes have been used to model this problem, as depicted in Fig. 24. A fixed enriched area with radius $r = 0.2a$ has been used for crack tip enrichment in all propagation steps. Kim and Paulino in [74] provided the critical load $P_{cr}$ only in the initial step. Values of $P_{cr}$, stress intensity factors and phase angle of SIFs at the initial step are compared in Table 12, which shows almost the same results. Also, the initial angle of crack propagation based on the maximum energy release rate criterion has been obtained 7.009° and 3.988° for $\zeta = 0.17$ and $\zeta = 0.58$, respectively, which are identical to the experimental values of 7° and 4°. These results have been obtained by a constant crack increment $\Delta a = 2$ mm in all propagation steps. The crack trajectory has been shown in Figs. 25 and 26 and compared with the experimental ones and those presented by Kim and Paulino [74]. Again, excellent agreement is observed between the XFEM predictions and reference results.

Fig. 27 compares the almost identical results obtained from the maximum hoop stress and maximum energy release rate propagation criteria on the crack trajectory. Also, critical values of $P_{cr}$ versus crack length ($P_{cr}$ in each increment) are depicted in Fig. 28. Again, similar results are expectedly obtained by the two propagation criteria. In order to study the effect of length of crack increment on crack trajectory and $P_{cr}$ two other crack propagation increments equal to $\Delta a = 1$ mm and $\Delta a = 3$ mm are considered. Figs. 29 and 30 show the effect of crack increment length on crack trajectory in comparison with available experimental and numerical results. The same results have been obtained for all increment lengths. It may be attributed to the fact that after an initial kink, the effect of mixed mode fracture is largely reduces to a single mode, which is less affected by crack increment length. The effect of $\Delta a$ on critical load has been depicted in Fig. 31, which shows identical critical loads for $\Delta a = 2$ mm and $\Delta a = 3$ mm but there is about 10% difference for propagation increment $\Delta a = 1$ mm.

Also, comparison of the effect of crack increment length in different crack propagation criteria has been depicted in Fig. 29, where identical results are observed.

Rousseau and Tippur [73] also modeled their specimen by ANSYS using 10,000 eight-noded elements and 30,000 nodes.

![Fig. 29.](image-url) Comparison of the effect of crack increment length on crack trajectory for XFEM and experimental one [73].

![Fig. 30.](image-url) Comparison of the effect of crack increment length on crack trajectory for XFEM and the finite element model by Kim and Paulino [74].

![Fig. 31.](image-url) Comparison of the effect of crack increment length on the critical load.

<table>
<thead>
<tr>
<th>$\zeta$</th>
<th>$E$ (MPa)</th>
<th>$v$</th>
<th>$K_{cr}$ (MPa $\sqrt{m}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3000</td>
<td>0.35</td>
<td>1.2</td>
</tr>
<tr>
<td>0.17</td>
<td>3300</td>
<td>0.34</td>
<td>2.1</td>
</tr>
<tr>
<td>0.33</td>
<td>5300</td>
<td>0.33</td>
<td>2.7</td>
</tr>
<tr>
<td>0.58</td>
<td>7300</td>
<td>0.31</td>
<td>2.7</td>
</tr>
<tr>
<td>0.83</td>
<td>8300</td>
<td>0.3</td>
<td>2.6</td>
</tr>
<tr>
<td>1</td>
<td>8600</td>
<td>0.29</td>
<td>2.6</td>
</tr>
</tbody>
</table>

Table 11
Modulus of elasticity, Poisson’s ratio and critical SIF in graded domain of glass-filled epoxy [73].
Large differences in the number of nodes and elements show the efficiency of present XFEM method. Also, Comi and Mariani [75] modeled this problem by XFEM method without using crack tip enrichments and obtained $\theta = 7.22^\circ$ for $\zeta = .17$ and $\theta = 4.07^\circ$ for $\zeta = .58$. They employed 13,084 three-node elements and 6610 nodes that are much higher than the present XFEM with crack tip enrichments (4848 elements and 2547 nodes).

To further investigate the crack propagation in orthotropic FGM media, the equivalent material properties, such as Eq. (83), are considered; where $\kappa_0 = .5$, $E$ and $v$ vary according to Table 11, and $\lambda = E_2/E_1$. Also, $K_{II}^{cr}$ is assumed to follow Table 11 and $K^{cr} = K_{II}^{cr}E_1/E_2$. Crack trajectories for $\lambda = 5, 1, 0.2$ and 0.01 are depicted in Fig. 32. It is clearly observed that by decreasing $\lambda$ (i.e. strong $x$ direction), crack has a tendency to propagate towards the $x$ direction.

Also, variations of load $P$ versus the crack length, after crack kinking, for different values of $\lambda$ are depicted in Fig. 33. It is clearly observed that by decreasing $\lambda$ (increasing ratio of $E_1/E_2$), the critical load $P$ increases, which is caused by higher effect of $\sigma_{xx}$ than $\sigma_{yy}$ in the present loading. It should be noted that for each point on the total crack length axis a different load $P$ is obtained, which corresponds to a different crack path, as depicted in Fig. 33.

### Table 12
Comparison of $P_{cr}$, stress intensity factors and phase angle of SIFs.

<table>
<thead>
<tr>
<th>$\zeta$</th>
<th>$P_{cr}$</th>
<th>$K_I$</th>
<th>$K_{II}$</th>
<th>$\psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ref. [74]</td>
<td>Present XFEM</td>
<td>Ref. [74]</td>
<td>Present XFEM</td>
<td>Ref. [74]</td>
</tr>
<tr>
<td>0.17</td>
<td>249.3</td>
<td>250</td>
<td>2.088</td>
<td>2.087</td>
</tr>
<tr>
<td>0.58</td>
<td>298</td>
<td>300</td>
<td>2.695</td>
<td>2.694</td>
</tr>
</tbody>
</table>

Fig. 32. Crack trajectories. (a) overall pattern for $\lambda = 5, 1, 0.2$ and (b) details for $\lambda = 0.2$.

Fig. 33. Variations of load $P$ versus crack length for different values of $\lambda$ (after crack kinking).
9. Conclusion

In this study, XFEM has been applied to orthotropic functionally graded materials. Orthotropic enrichments have been employed to accurately represent singular stress field near a crack tip. The stress intensity factors have been calculated from the incompatible interaction integral. It has been illustrated that orthotropic XFEM needs far fewer DOFs than conventional FEM to achieve the same level of accuracy. In addition, high convergence rates are obtained by this method. Three cases of tip enrichment (isotropic enrichment, orthotropic enrichment and no enrichment) have been compared for orthotropic FGM problems and shown that for the same mesh configuration and contour integral, orthotropic enrichments yield far accurate results. Also, simulation of crack propagation in an isotropic problem showed good agreement with the experimental results. It has also been extended and numerically discussed for propagation in orthotropic FGM media.

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References

[8] M. Kasmallari. Surface and internal crack problems in a homogeneous substrate coated by a graded layer, Ph.D. dissertation, Department of Mechanical Engineering and Mechanics, Lehigh University, Bethlehem, PA, USA; 1996.