

ORIGINAL ARTICLE



Evaluation of T-stress in stationary and propagating adiabatic cracks in FGM subjected to thermo-mechanical loading

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ABSTRACT

The aim of this paper is to study the T-stress and its effect on crack propagation in FGM materials with adiabatic cracks subjected to thermo-mechanical loading based on the J-integral formulation and the extended finite element method (XFEM). Path-independent J-integral responses of T-stress for adiabatic cracks and the effects of incompatible terms for FGMs are discussed. Having examined the T-stress parameter, the effect of fracture characteristic/process length on crack propagation angle in adiabatic cracks is studied. The results show that the initial crack propagation angle deviates from the standard propagation angle by increasing the size of the fracture process zone.

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T-stress; adiabatic crack; extended finite element method (XFEM); FGM; interaction integral; crack propagation

1. Introduction

Functionally graded materials (FGMs) are growingly applied in many engineering fields, such as medical and aerospace industries, to overcome the shortcomings of homogeneous materials in handling both the mechanical and thermal effects, or potential inconsistencies that may exist around the interfaces of layered composites [1, 2]. One of the main applications of FGMs is in the conditions where both the mechanical and thermal strength of materials are simultaneously required, especially in thermal barrier coating. While traditional layered ceramic/metals materials could severely suffer in thermo-mechanical conditions and experience delamination due to the sudden changes of thermal properties between different layers [3], FGMs could substantially improve the response by providing a continuous variation of material properties. Nevertheless, thermal fracture analysis of FGMs remains an important task, due to the existing initial cracks during the complex manufacturing process and the severe thermo-mechanical conditions.

In a thermo-mechanical fracture analysis, crack surfaces play an important role in the thermal response of the material [4]. The thermal condition of a crack can be specified by the thermal insulation coefficient, which is defined across the crack surfaces [4, 5]. The perfect conduction and the perfect insulation (zero-heat flux) are the two extreme cases of thermal crack analysis [5]. In practical applications, when an air-gap exists in between the crack surfaces, the insulated crack assumption, also known as the adiabatic crack, may be adopted. From the analytical perspective, the heat conduction field becomes singular near the crack tips of an adiabatic crack. This has to be properly considered in numerical

simulations. A comprehensive survey on literature shows that most available literature on thermal analysis of FGMs are dedicated to the simpler case of nonsingular condition [6–10], and only very limited researches are devoted to the study of singular adiabatic cracks [5, 11, 12]. Accordingly, the main focus of the present paper is set to study of adiabatic cracks in FGMs.

In fracture mechanics, the concepts of stress intensity factors (SIFs) and T-stress are the main parameters that, respectively, reflect the singular and nonsingular responses of stress near a crack tip. Different numerical methods have been developed in recent years to study SIFs in FGMs, including the boundary element method [13–17], the finite element method [6, 12, 18–35], numerical manifold method [11, 36], graded finite element [37], peridynamic [3, 38], local Petrov–Galerkin method [39], and the extended finite element method (XFEM) [40–50]. Among them, XFEM has become one of the robust and effective numerical methods for analysis of fracture problems in isotropic and anisotropic functionally graded materials subject to general mechanical and thermal loadings. Due to the main advantages of XFEM in modeling cracks independent of mesh configuration, and reproducing the crack tip singular stress field, XFEM is adopted in the present study for fracture evaluation and crack propagation analysis of FGMs under thermo-mechanical loadings. The path-independent J-integral, well-established as a robust method to extract fracture parameters of FGMs under mechanical and thermo-mechanical loadings [6–8, 51–54], is adopted in this research.

On the other hand, few researchers have worked on evaluation of T-stress in FGMs under the thermo-mechanical loading [10, 11], but without considering its effect on crack

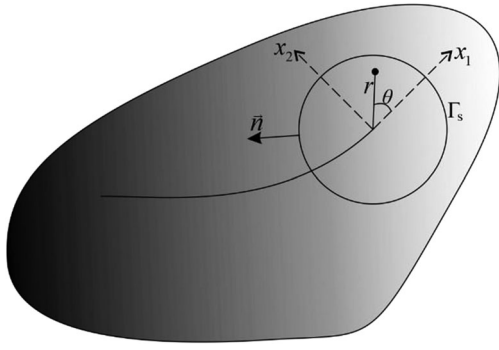


Figure 1. Definitions of the contour J-integral.

propagation angle. Accordingly, the present investigation is devoted to evaluation of T-stress in adiabatic cracks in FGMs under thermo-mechanical loadings based on the interaction integral method. The effect of T-stress in crack propagation is also explored within the context of XFEM methodology.

To this end, the theoretical basics of evaluation of fracture parameters of FGM materials, based on the interaction integral method, and the crack propagation criterion are reviewed in sections 2-5. Afterwards, the corresponding XFEM formulation is briefly explained in sections 6 and 7. In section 8, different thermal crack examples are numerically investigated, and the method is then employed to investigate a number of adiabatic crack problems in details.

2. FGM stress-strain relationship in thermo-mechanical condition

Considering Θ , ε , σ , E , G , ν and α as temperature, strain, stress, modulus of elasticity, shear modulus, Poisson's ratio and thermal expansion coefficient, the stress-strain relationship for plane stress problems in thermo-mechanical conditions can be written as,

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \alpha \Delta \Theta \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \quad (1)$$

where

$$G = \frac{E}{2(1+\nu)} \quad (2)$$

and ε_{ij} is the total strain, including the mechanical (ε_{ij}^m) and thermal (ε^{Θ}) strains, and $\alpha \Delta \Theta$ is the thermal strain ε^{Θ} . Clearly, the mechanical strain $\varepsilon_{ij}^m = \varepsilon_{ij} - \alpha_{ij} \Delta \Theta$ is related to the strain energy.

In the plane strain condition, the stress-strain relation is written as

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1-\nu^2}{E} & -\frac{\nu(1+\nu)}{E} & 0 \\ -\frac{\nu(1+\nu)}{E} & \frac{1-\nu^2}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \alpha(1+\nu) \Delta \Theta \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \quad (3)$$

where the last term represents the thermal strain in the plane strain condition. Notably, the total and thermal strains in the plane strain condition in the x_3 direction are

$$\varepsilon_{33} = 0, \quad \varepsilon_{33}^{\Theta} = \alpha \Delta \Theta \quad (4)$$

Accordingly, the mechanical strain in x_3 direction can be derived

$$\varepsilon_{33}^m = -\alpha \Delta \Theta \quad (5)$$

which must be considered in computing the strain energy for SIF and T-stress calculations.

3. SIFs and T-stress

Considering singular and nonsingular terms in the stress field around the crack tip, the analytical solutions for stresses components in the polar coordinate system can be written as [55]

$$\begin{aligned} \sigma_{rr} &= K_I f_{K_I}^{rr}(r, \theta) + K_{II} f_{K_{II}}^{rr}(r, \theta) + T f_T^{rr}(r, \theta) + \dots \\ \sigma_{\theta\theta} &= K_I f_{K_I}^{\theta\theta}(r, \theta) + K_{II} f_{K_{II}}^{\theta\theta}(r, \theta) + T f_T^{\theta\theta}(r, \theta) + \dots \\ \sigma_{r\theta} &= K_I f_{K_I}^{r\theta}(r, \theta) + K_{II} f_{K_{II}}^{r\theta}(r, \theta) + T f_T^{r\theta}(r, \theta) + \dots \end{aligned} \quad (6)$$

where $f_{K_I}^{rr}, f_{K_{II}}^{rr}, f_{K_I}^{\theta\theta}, f_{K_{II}}^{\theta\theta}, f_{K_I}^{r\theta}, f_{K_{II}}^{r\theta}$ are the functions representing the singular stress field around crack tips, and $f_T^{rr}, f_T^{\theta\theta}, f_T^{r\theta}$ are nonsingular higher-order functions. K_I and K_{II} are SIFs that explore the effects of geometry, boundary and loading conditions on singular-terms of the stress field, and T represents the T-stress parameter that is related to the higher-order terms of stress.

Different methods have been developed to compute SIFs and T-stress [7–10, 12, 51–53, 56–58]. Among them, the path-independent J-integral formulation, frequently employed by many researchers, is adopted in this work. Other methods include the displacement extrapolation [59], and the crack tip opening angle/displacement [60, 61].

4. J and interaction integrals

The original definition of the J-integral can be written as,

$$J = \lim_{\Gamma_s \rightarrow 0} \int_{\Gamma_s} (w \delta_{1j} - \sigma_{ij} u_{i,1}) n_j d\Gamma \quad (7)$$

where $w = 1/2 \varepsilon_{ij}^m \sigma_{ij}$ is the strain energy density and u_i is displacement field. Other parameters are depicted in Figure 1. ε_{33}^m and σ_{33} are non-zero in the plane strain cases and should be considered in the computation of strain energy.

The equivalent domain integral form of the J-integral can be written in terms of the ramp function Q (as depicted in Figure 2) [53],

$$\begin{aligned} J &= \int_A \left\{ \sigma_{ij} u_{i,1} - \left(\frac{1}{2} \sigma_{rs} \varepsilon_{rs}^m \right) \delta_{1j} \right\} Q_{,j} dA \\ &+ \int_A \left\{ \sigma_{ij} u_{i,1} - \left(\frac{1}{2} \sigma_{rs} \varepsilon_{rs}^m \right) \delta_{1j} \right\} Q dA \end{aligned} \quad (8)$$

where Q is a smooth function varying from 1 at the crack tip to zero at the outer edge of the integral domain. Q is

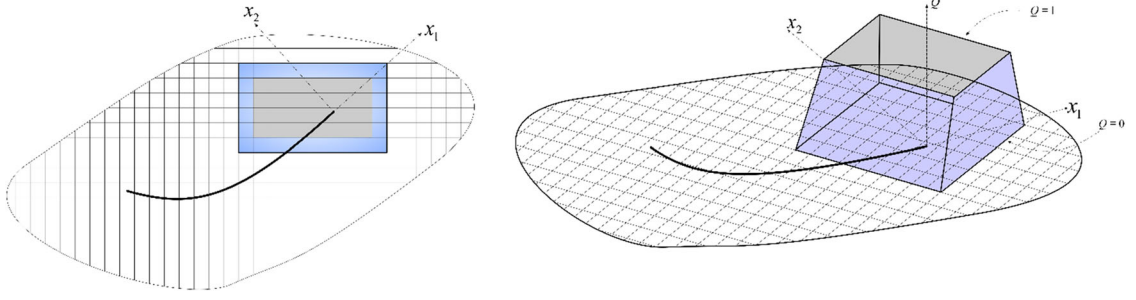


Figure 2. Definition of smooth function Q in the equivalent domain integral method.

usually discretized by the standard shape functions of FEM, as depicted in Figure 2.

4.1. Incompatible M-integral

Using a compatible auxiliary displacement field around the crack tip (denoted by u_i^{aux}) and the superposition assumption, allows for deriving the J-integral for a combined effect of actual and auxiliary fields:

$$J = J^{act} + J^{aux} + M \quad (9)$$

where J^{act} and J^{aux} are obtained by replacing the actual and auxiliary fields in Eq. (8), respectively [53]. M is called the interaction integral that includes the coupled effect of actual and auxiliary fields [53]:

$$M = \int_A \left\{ \left(\sigma_{ij}^{act} u_{i,1}^{aux} + \sigma_{ij}^{aux} u_{i,1}^{act} \right) - \left(\sigma_{rs}^{act} e_{rs}^{aux} \right) \delta_{1j} \right\} Q_{,j} dA + \int_A \left\{ \left(\sigma_{ij}^{act} u_{i,1}^{aux} + \sigma_{ij}^{aux} u_{i,1}^{act} \right) - \left(\sigma_{rs}^{act} e_{rs}^{aux} \right) \delta_{1j} \right\}_{,j} Q dA \quad (10)$$

where superscripts *act* and *aux* refer to the actual and auxiliary fields. The actual fields (displacement, strain and stress) are calculated from the main problem by XFEM, whereas the auxiliary fields are extracted from the analytical solutions, as will be discussed in the next section.

According to the M-integral formulation, the auxiliary fields need to be evaluated in different locations inside the domain integral. On the other hand, the auxiliary field that is conventionally used in the M-integral computation is based on the asymptotic analytical solutions around a crack tip based on a homogeneous material. Consequently, the M-integral formulation can lead to some inconsistency that should be considered in formulations. Three approaches have well been implemented in the literature [8], namely, incompatible, non-equilibrium and constant constitutive formulations. Among them, the incompatible formulation has been frequently used, especially in thermo-mechanical conditions [9, 53]. In the incompatible formulation, the auxiliary stress and displacement fields are directly adopted from the analytical solutions computed by the mechanical properties at the crack tip location. Then, the strain field is calculated directly from $\epsilon_{ij}^{aux} = s_{ijkl} \sigma_{kl}^{aux}$, where s_{ijkl} is the compliance tensor. Consequently, the compatibility condition can no longer be satisfied in the domain:

$$\epsilon_{ij}^{aux} \neq 1/2(u_{i,j}^{aux} + u_{j,i}^{aux}) \quad (11)$$

The effect of this inconsistency in the M-integral formulation is compensated to some extent by adding the term $(s_{ijmn})_{tip} - s_{ijmn}$ [9, 53]:

$$M = \int_A \left\{ \left(\sigma_{ij}^{act} u_{i,1}^{aux} + \sigma_{ij}^{aux} u_{i,1}^{act} \right) - \left(\sigma_{rs}^{act} e_{rs}^{aux} \right) \delta_{1j} \right\} Q_{,j} dA + \int_A \left\{ \sigma_{ij}^{act} \left[(s_{ijmn})_{tip} - s_{ijmn} \right] \sigma_{mn,1}^{aux} + \sigma_{ij}^{aux} e_{ij,1}^{act,h} \right\} Q dA \quad (12)$$

where $\epsilon_{ij,1}^{act,h}$ is defined for plane stress problems as:

$$\epsilon_{ij,1}^{act,h} = (\alpha_{,1} \Delta \Theta + \alpha \Delta \Theta_{,1}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (13)$$

and for plane strain conditions:

$$\epsilon_{ij,1}^{act,h} = (\alpha_{,1} (1 + \nu) \Delta \Theta + \alpha \nu_{,1} \Delta \Theta + \alpha (1 + \nu) \Delta \Theta_{,1}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (14)$$

4.2. Auxiliary fields for FGM to evaluate SIFs

Asymptotic analytical solutions are used as the auxiliary fields in the M-integral formulations. Available analytical solutions are mainly based on homogeneous material formulations and may violate some basic conditions in FGM materials. In the incompatible formulation, however, the following homogeneous displacement, stress and strain auxiliary fields are adopted [8, 59]:

$$u_1^{aux} = K_I^{aux} \left[\frac{1}{2\mu_{tip}} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left(\kappa_{tip} - 1 + 2 \sin^2 \frac{\theta}{2} \right) \right] + K_{II}^{aux} \left[\frac{1}{2\mu_{tip}} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left(\kappa_{tip} + 1 + 2 \cos^2 \frac{\theta}{2} \right) \right] \\ u_2^{aux} = K_I^{aux} \left[\frac{1}{2\mu_{tip}} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left(\kappa_{tip} + 1 - 2 \cos^2 \frac{\theta}{2} \right) \right] + K_{II}^{aux} \left[\frac{1}{2\mu_{tip}} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left(\kappa_{tip} - 1 - 2 \sin^2 \frac{\theta}{2} \right) \right] \quad (15)$$

$$\begin{aligned}
\sigma_{11}^{aux} &= K_I^{aux} \left[\frac{1}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \right] \\
&\quad + K_{II}^{aux} \left[\frac{-1}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \left(2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right) \right] \\
\sigma_{22}^{aux} &= K_I^{aux} \left[\frac{1}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \right] \\
&\quad + K_{II}^{aux} \left[\frac{1}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right] \\
\sigma_{12}^{aux} &= \sigma_{21}^{aux} = K_I^{aux} \left[\frac{1}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right] \\
&\quad + K_{II}^{aux} \left[\frac{1}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \right]
\end{aligned} \quad (16)$$

$$\varepsilon_{ij}^{aux} = s_{ijkl} \sigma_{kl}^{aux} \quad (17)$$

where μ_{tip} is the shear modulus and κ_{tip} is the material constant defined as (computed at the position of crack tip)

$$\begin{aligned}
\text{plane stress : } \kappa_{tip} &= \frac{3 - \nu_{tip}}{1 + \nu_{tip}} \\
\text{plane strain : } \kappa_{tip} &= 3 - 4\nu_{tip}
\end{aligned} \quad (18)$$

It is re-iterated that ε_{ij}^{aux} cannot be computed directly from the analytical auxiliary field in an incompatible formulation and should be calculated from Eq. (17) with material properties s_{ijkl} at an appropriate location (e.g. Gauss points in FEM).

4.3. Auxiliary fields for FGM to evaluate T-stress

A similar procedure to SIF computation is used to obtain the T-stress for adiabatic cracks by the interaction integral method. The main difference is related to the adopted auxiliary field. In the T-stress case, the analytical asymptotic field is obtained from the solution of a point load parallel to the crack applied at the crack tip location, as depicted in Figure 3. In this case, a singular field of order $1/r$ is derived in the stress field, which helps to evaluate the nonsingular effect of T-stress.

With a similar incompatible strain formulation used in SIF calculations, the auxiliary fields for computation of the T-stress are [6, 52]:

$$u_1^{aux} = -\frac{f(\kappa_{tip} + 1)}{8\pi\mu_{tip}} \ln \frac{r}{d} - \frac{f}{4\pi\mu_{tip}} \sin^2 \theta \quad (19)$$

$$u_2^{aux} = -\frac{f(\kappa_{tip} - 1)}{8\pi\mu_{tip}} \theta + \frac{f}{4\pi\mu_{tip}} \sin \theta \cos \theta$$

$$\begin{aligned}
\sigma_{11}^{aux} &= -\frac{f}{\pi r} \cos^3 \theta \\
\sigma_{22}^{aux} &= -\frac{f}{\pi r} \cos \theta \sin^2 \theta
\end{aligned} \quad (20)$$

$$\begin{aligned}
\sigma_{12}^{aux} &= \sigma_{21}^{aux} = \frac{f}{\pi r} \cos^2 \theta \sin \theta \\
\varepsilon_{ij}^{aux} &= s_{ijkl} \sigma_{kl}^{aux}
\end{aligned} \quad (21)$$

where f is a point load applied at the crack tip location in parallel with the crack line as depicted in Figure 3, and d is the x_1 component of a fixed point [7].

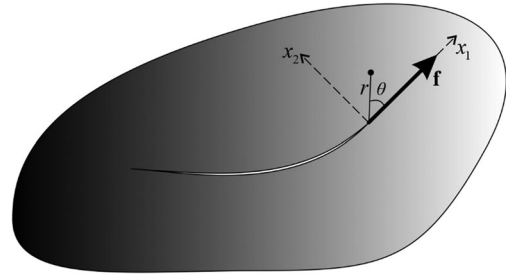


Figure 3. The basic problem of a point load at the crack tip and along the crack line, as a basis to find the analytical solution to evaluate the T-stress.

4.4. SIF computation

Replacing the auxiliary fields from Eqs. (15–17) into (12), the M-integral can be expressed in terms of the actual and auxiliary SIFs as [6]

$$M = \frac{2}{E_{tip}^*} (K_I K_I^{aux} + K_{II} K_{II}^{aux}) \quad (22)$$

where

$$E_{tip}^* = \begin{cases} E_{tip} & : \text{plane stress} \\ E_{tip}^* = \frac{E_{tip}}{1 - \nu_{tip}^2} & : \text{plane strain} \end{cases} \quad (23)$$

Finally, the actual SIFs are then calculated from relation (22) as

$$\begin{aligned}
K_I &= \frac{E_{tip}^*}{2} M^{(1)} & \text{where : } M^{(1)} &= f(K_I^{aux} = 1, K_{II}^{aux} = 0) \\
K_{II} &= \frac{E_{tip}^*}{2} M^{(2)} & \text{where : } M^{(2)} &= f(K_I^{aux} = 0, K_{II}^{aux} = 1)
\end{aligned} \quad (24)$$

4.5. T-Stress computation

Replacing the auxiliary fields (19–21) in (12), leads to computation of T-stress [6]

$$T = \frac{E_{tip}^* M}{f} - \vartheta_{tip} \alpha_{tip} \Delta \Theta_{tip} E_{tip}^* \quad (25)$$

where

$$\vartheta_{tip} = \begin{cases} 1 & : \text{plane stress} \\ 1 + \nu_{tip} & : \text{plane strain} \end{cases} \quad (26)$$

5. Crack propagation angle

The generalized maximum hoop stress criterion as described in [7], is used to find the crack propagation angle. Here, the T-stress values are included in the formulation. Stress components in a polar coordinate system in terms of SIFs and T-stress are expressed as [7, 52]

$$\sigma_{rr} = \frac{1}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[K_I \left(1 + \sin^2 \frac{\theta}{2} \right) + \frac{3}{2} K_{II} \left(\sin \theta - 2 \tan \frac{\theta}{2} \right) \right] + T \cos^2 \theta \quad (a)$$

$$\sigma_{\theta\theta} = \frac{1}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[K_I \cos^2 \frac{\theta}{2} - \frac{3}{2} K_{II} \sin \theta \right] + T \sin^2 \theta \quad (b)$$

$$\sigma_{r\theta} = \frac{1}{2\sqrt{2\pi r}} \cos \frac{\theta}{2} [K_I \sin \theta + K_{II} (3 \cos \theta - 1)] - T \sin \theta \cos \theta \quad (c)$$

(27)

where K_I , K_{II} and T are mode I, mode II and T-stress parameters, respectively. According to the maximum hoop

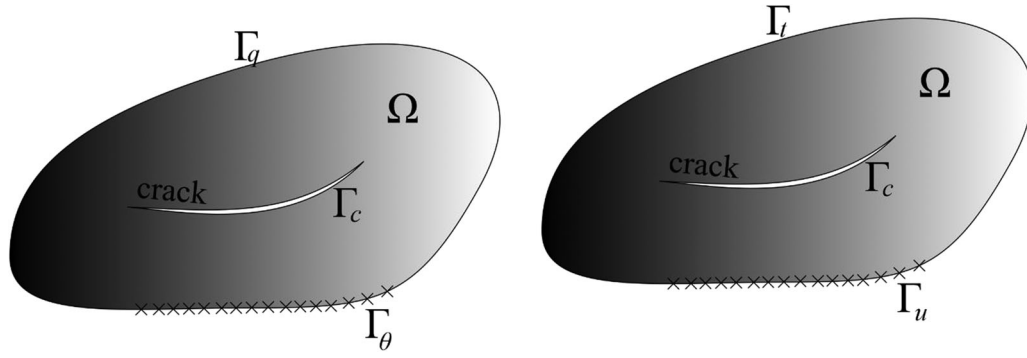


Figure 4. Domain and boundary conditions in the thermal adiabatic crack (left) and the mechanical equilibrium (right) problems.

stress criterion, a crack propagates in the direction in which the hoop stress $\sigma_{\theta\theta}$ becomes maximum. Accordingly, the crack propagation angle can be calculated by maximizing

$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0 \Rightarrow \theta = \theta_p \quad (28)$$

where θ_p is the crack propagation angle and is calculated from [7, 52, 55]

$$\cos \frac{\theta_p}{2} \left[K_I \sin \theta_p + K_{II} (3 \cos \theta_p - 1) - \frac{16}{3} T \sqrt{2\pi r_c} \sin \frac{\theta_p}{2} \cos \theta_p \right] = 0 \quad (29)$$

Solution of this equation leads to calculation of the crack propagation angle θ_p . A characteristic crack length scale r_c and the effect of the T-stress terms have been introduced in relation (29). r_c is related to the size of the fracture process zone [52]. It is discussed in [62, 63] that if $r \rightarrow 0$, the higher order terms in the stress field vanish. Therefore, the crack propagation criterion should be evaluated at the critical distance r_c , in order to take into account the effect of higher order terms of stress. r_c is a material property [55] that should be generally determined by experimental tests [62]. Setting $r_c = 0$ leads to the standard maximum hoop stress criterion (without the T-stress terms):

$$\cos \frac{\theta_p}{2} [K_I \sin \theta_p + K_{II} (3 \cos \theta_p - 1)] = 0 \quad (30)$$

After computing the crack propagation angle θ_p , the crack propagation criterion can be written by replacing θ with θ_p and r with r_c in $\sigma_{\theta\theta}$ (relation 27-b) [7]

$$\sqrt{2\pi r_c} (\sigma_{\theta\theta})_c = K_{Ic} = \cos \frac{\theta_p}{2} \left[K_I \cos^2 \theta_p - \frac{3}{2} K_{II} \sin \theta_p \right] + \sqrt{2\pi r_c} T \sin^2 \theta_p \quad (31)$$

It should be noted that K_{Ic} is generally variable in FGM materials [53], and a common assumption is to consider it changing with the same function as of the modulus of elasticity [53]. The effect of the characteristic crack length scale r_c on crack propagation angle is investigated in detail in numerical examples.

6. Governing steady-state thermo-mechanical equations of an adiabatic crack

A steady state thermo-mechanical formulation is adopted in this section.

6.1. Steady state thermal formulation

The following steady-state thermal equation is solved in the first step [54].

$$\begin{aligned} q_{i,i} + s_h &= 0 & \text{on } \Omega \\ \Theta &= \bar{\Theta} & \text{on } \Gamma_\theta \\ q_n &= \bar{q} & \text{on } \Gamma_q \end{aligned} \quad (32)$$

where s_h is the heat source and the heat flux q_i is defined as

$$q_i(x_1, x_2) = k(x_1, x_2) \Theta_{,i}(x_1, x_2) \quad (33)$$

$\bar{\Theta}$ and $\bar{q}$ are the prescribed temperature and the heat flux on Γ_θ and Γ_q , respectively, and k is the heat conductivity that varies within the non-homogeneous domain. Ω is the domain of the problem with boundary $\Gamma = \Gamma_\theta \cup \Gamma_q$. Note that the crack faces Γ_c , as depicted in Figure 4, is part of Γ_q in adiabatic cracks, and the heat flux $\mathbf{q}$ is zero across the crack faces. Note that in non-adiabatic crack problems, Γ_c is only considered in solving equilibrium equation, without any effect (neither Γ_θ nor Γ_q) on thermal equation.

6.2. Equilibrium equation

After solving the thermal equation to find the temperature field Θ , the following equilibrium equation should be solved

$$\begin{aligned} \sigma_{ij,j} + b_i &= 0 & \text{on } \Omega \\ u &= \bar{u} & \text{on } \Gamma_u \\ \sigma_{ij} n_j &= \bar{t}_i & \text{on } \Gamma_t \end{aligned} \quad (34)$$

This equation is related to the thermal equation by the definition of stress, as described in relations (1) and (3). Here, $\bar{u}$ and $\bar{t}$ are the prescribed displacement and tractions on Γ_u and Γ_t , respectively. $\Gamma = \Gamma_u \cup \Gamma_t$ is the boundary of the problem, and crack faces is part of Γ_t (Figure 4).

7. XFEM for adiabatic thermo-mechanical problems

XFEM, as a well-established method in fracture mechanics, is used in the present study to solve the present thermo-

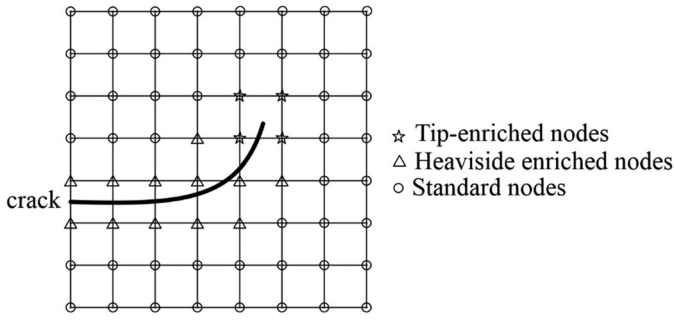


Figure 5. Definition of sets of standard, Heaviside and tip-enriched nodes in XFEM formulation.

mechanical equations of fractured media. The main capability of XFEM is to introduce crack faces as discontinuous fields via additional enrichment functions in field variables (instead of geometrical modeling of crack faces). Additionally, XFEM is capable of reproducing singular stress fields around the crack tip [60, 64, 65].

Considering that the crack faces act as a boundary in adiabatic cracks, XFEM enrichment procedure should be used in both thermal and mechanical parts.

7.1. XFEM formulation

Generally, the displacement field in XFEM formulation is composed of the standard and enrichment parts. The standard approximation is based on the standard FEM shape functions, regardless of any crack in the domain. The enrichment part of approximation is to represent the discontinuity or singular responses. Accordingly, the approximation in mathematical form is expressed as [59]

$$u = u^{st} + u^{en} \quad (35)$$

where u^{st} and u^{en} refer to the standard and enrichment parts of the approximation, respectively. The enriched part is divided into the Heaviside to describe the discontinuity of field variables (displacement and temperature) across crack faces, and the tip enrichment to reproduce the singular responses of stress and heat fluxes near the crack tip. Accordingly, u^{en} is written as [59]

$$u^{en} = u^{Heaviside} + u^{tip} \quad (36)$$

where

$$u^{Heaviside} = \sum_{i \in n^H} N_i(\mathbf{x}) [H(\zeta) - H(\zeta_i)] b_i^{u-H} \quad (37)$$

$$u^{tip} = \sum_{i \in n^{tip}} N_i(\mathbf{x}) \left(\sum_{j=1}^{NTF} [F_j^u(\mathbf{x}) - F_j^u(\mathbf{x}_i)] b_{ij}^{u-T} \right) \quad (38)$$

b_i^{u-H} and b_{ij}^{u-T} are additional degrees of freedom related to the Heaviside and tip enrichments, respectively, $N_i(\mathbf{x})$ is the standard shape functions of the FEM, and n^H and n^{tip} are the set of nodes that are enriched by the Heaviside and tip enrichments, respectively, as depicted in Figure 5. $F_j^u(\mathbf{x})$ are the tip-enrichment functions that will be described in the next section for different conditions, and $H(\zeta)$ is the discontinuous Heaviside function

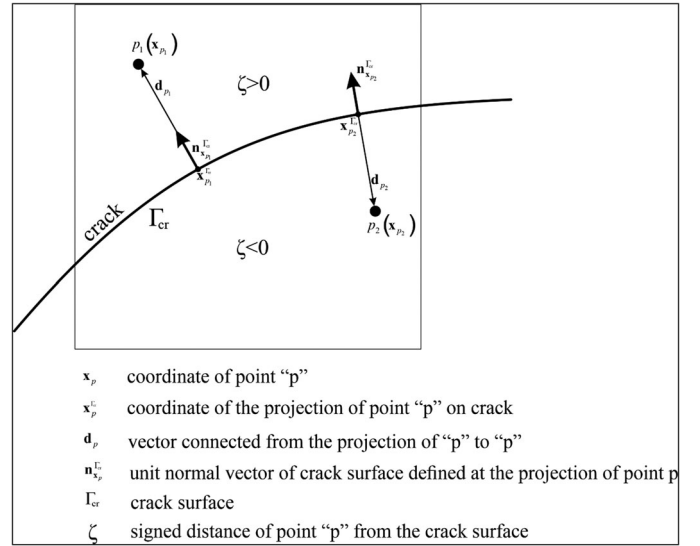


Figure 6. Definition of Heaviside enrichments in XFEM formulation.

$$H(\zeta) = \begin{cases} 1 & \forall \zeta > 0 \\ -1 & \forall \zeta < 0 \end{cases} \quad (39)$$

$$\zeta(\mathbf{x}) = \mathbf{d} \cdot \mathbf{n}_x^{\Gamma_{cr}} \quad (40)$$

Parameters $\zeta(\mathbf{x}) = \mathbf{d} \cdot \mathbf{n}_x^{\Gamma_{cr}}$ and $\zeta(\mathbf{x}) = \mathbf{d} \cdot \mathbf{n}_x^{\Gamma_{cr}}$ are defined in Figure 6. In relations (37) and (38), $H(\zeta_i)$ and $F_j^u(\mathbf{x}_i)$ are values of the Heaviside and tip enrichments at node i .

The same procedure is used to approximate the temperature field in adiabatic cracks as follows

$$\Theta = \Theta^{st} + \Theta^{en} \quad (41)$$

$$\Theta^{en} = \Theta^{Heaviside} + \Theta^{tip} \quad (42)$$

$$\Theta^{Heaviside} = \sum_{i \in n^H} N_i(\mathbf{x}) [H(\zeta) - H(\zeta_i)] b_i^{\Theta-H} \quad (43)$$

$$\Theta^{tip} = \sum_{i \in n^{tip}} N_i(\mathbf{x}) [F^\theta(\mathbf{x}) - F^\theta(\mathbf{x}_i)] b_i^{\Theta-T} \quad (44)$$

The general concept of all parameters are the same as displacement field approximation with the following definitions:

- $b_i^{\Theta-H}$ additional DOFs related to the Heaviside enrichment functions of the temperature field
- $b_i^{\Theta-T}$ additional DOFs related to the tip enrichment functions of the temperature field
- $N_i(\mathbf{x})$ standard shape functions of FEM
- $F^\theta(\mathbf{x})$ tip enrichment functions of temperature at point x (usually a Gauss point)
- $F^\theta(\mathbf{x}_i)$ tip enrichment functions of temperature at node i
- $H(\zeta)$ heaviside functions at point $x(\zeta)$ (usually a Gauss point)
- $H(\zeta_i)$ heaviside functions at node i

Note that the geometric approximation of XFEM formulation is exactly the same as the standard FEM

$$\mathbf{x} = \sum_{i \in n^{st}} N_i \mathbf{x}_i \quad (45)$$

where n^{st} is the set of standard nodes of finite element model. It should be stated that in the non-adiabatic cracks,

where crack faces are not part of essential and natural boundaries, the temperature field approximation becomes similar to the standard FEM interpolation as $\Theta = \Theta^{st}$.

7.2. Tip enrichment functions

Enrichment functions are generally derived from the available analytical asymptotic solutions in fracture mechanics. They can be different based on the type of problem: i.e. isotropic problems [59], orthotropic cases [66] and plate and shells [60]. In the present study, the isotropic enrichment functions for the displacement field are used [59]

$$F^u(\mathbf{x}) = \left\{ \sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \frac{\theta}{2} \sin \theta, \sqrt{r} \cos \frac{\theta}{2} \sin \theta \right\} \quad (46)$$

Among these enrichments, $\sqrt{r} \sin \theta/2$ is more important because it also guarantees the discontinuity of field variable in the enriched element.

A single tip enrichment function is used for the temperature field [53]

$$F^\theta(\mathbf{x}) = \sqrt{r} \sin \frac{\theta}{2} \quad (47)$$

7.3. Discretization

Considering the displacement and temperature approximations and enrichment functions in sections 7.2 and 7.3, the Discretized matrix form of the thermal and equilibrium equations are briefly discussed in this section.

7.3.1. Discretization of the thermal equation

The final discretized form of the thermal equation can be expressed as [53]

$$\mathbf{K}^\Theta \Theta + \mathbf{f}^\Theta = 0 \quad (48)$$

where the extended nodal temperature vector Θ is defined based on the standard nodal Θ^{st} and enriched Θ^{en} unknowns as

$$\Theta = \{ \Theta^{st} \quad \Theta^{en} \} \quad (49)$$

The matrix $\mathbf{K}^\Theta$ and vector $\mathbf{f}^\Theta$ are assembled from each node i :

$$\mathbf{K}_i^\Theta = \int_{\Omega} \left(\mathbf{B}_i^\Theta \right)^T k \left(\mathbf{B}_i^\Theta \right) d\Omega \quad (50)$$

$$\mathbf{f}_i^\Theta = \int_{\Gamma_q} \left(\mathbf{N}_i^\Theta \right)^T \bar{q} d\Gamma \quad (51)$$

where $\mathbf{N}_i^\Theta$ and $\mathbf{B}_i^\Theta$ are the matrix of temperature shape functions and their derivatives, respectively, as defined by

$$\mathbf{N}_i^\Theta = \begin{cases} N_i & \text{for standard nodes} \\ \begin{bmatrix} N_i & N_i(H(\zeta) - H(\zeta_i)) \end{bmatrix} & \text{for heaviside enriched nodes} \\ \begin{bmatrix} N_i & N_i(F^\theta(\mathbf{x}) - F^\theta(\mathbf{x}_i)) \end{bmatrix} & \text{for tip enriched nodes} \end{cases} \quad (52)$$

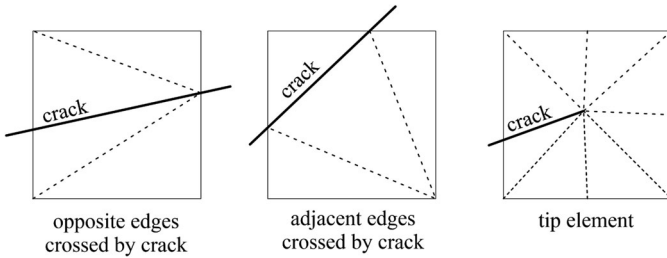


Figure 7. The sub-triangulation technique to compute integrals inside the enriched elements.

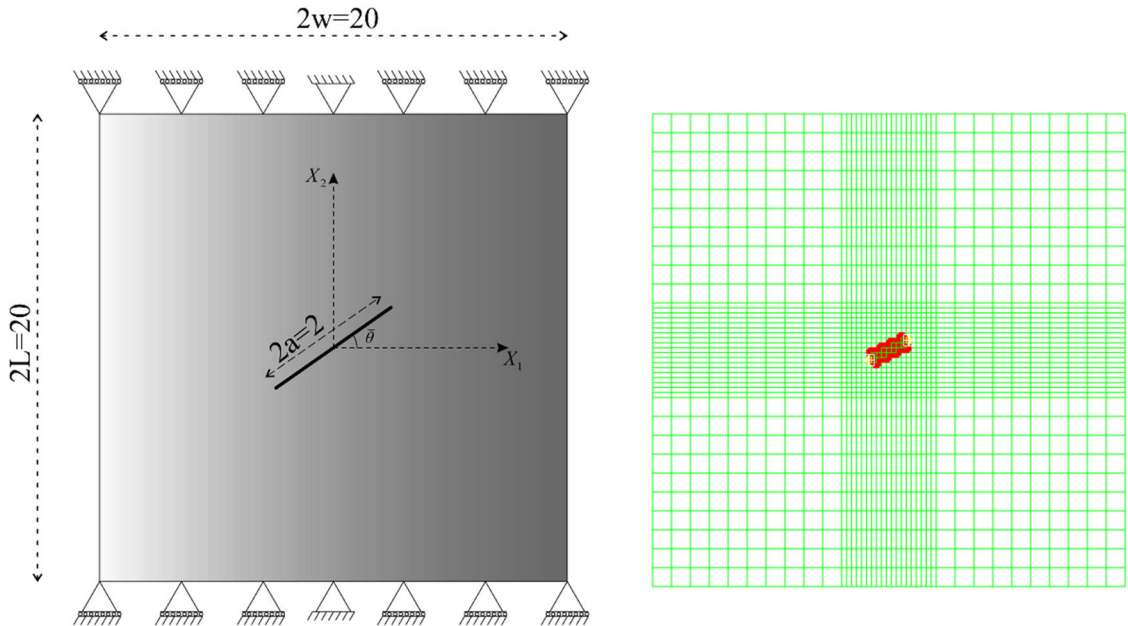


Figure 8. Geometry, boundary conditions and mesh configuration of the mixed-mode crack under thermal loading.

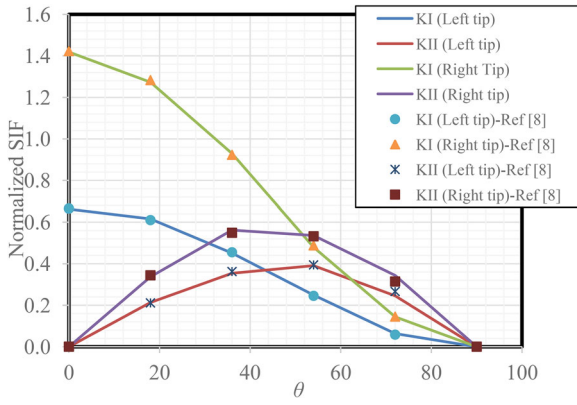


Figure 9. Comparison of the normalized K_I and K_{II} stress intensity factor for the cracks with different angles ($\beta = 0.5$).

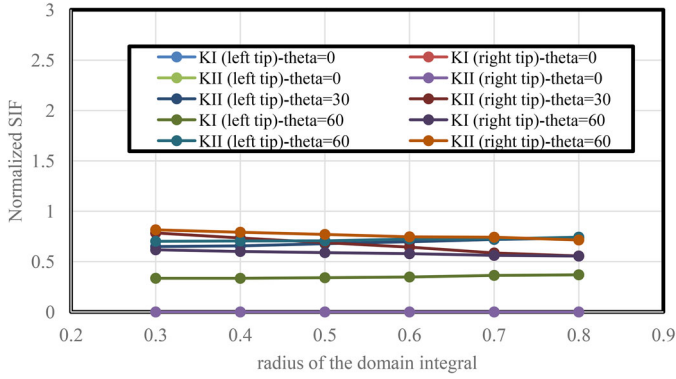


Figure 10. Variation of normalized K_I and K_{II} for crack angles 0° and 60° ($\beta = 0.5$).

$$\mathbf{B}_i^\Theta = \begin{cases} \begin{bmatrix} N_{i,x_1} \\ N_{i,x_2} \end{bmatrix} & \text{for standard nodes} \\ \begin{bmatrix} N_{i,x_1} & (N_i(H(\zeta) - H(\zeta_i)))_{,x_1} \\ N_{i,x_2} & (N_i(H(\zeta) - H(\zeta_i)))_{,x_2} \end{bmatrix} & \text{for heaviside enriched nodes} \\ \begin{bmatrix} N_{i,x_1} & (N_i(F^\theta(\mathbf{x}) - F^\theta(\mathbf{x}_i)))_{,x_1} \\ N_{i,x_2} & (N_i(F^\theta(\mathbf{x}) - F^\theta(\mathbf{x}_i)))_{,x_2} \end{bmatrix} & \text{for tip enriched nodes} \end{cases} \quad (53)$$

In non-adiabatic cases, Θ^{en} is omitted and the aforementioned discretization is reduced to the standard FEM formulation. Accordingly, the standard FEM shape functions and their derivatives are used in Eqs. (52) and (53) (i.e. all nodes are considered as the standard nodes):

$$\mathbf{N}_i^\Theta = N_i \quad (54)$$

$$\mathbf{B}_i^\Theta = \begin{bmatrix} N_{i,x_1} \\ N_{i,x_2} \end{bmatrix} \quad (55)$$

7.3.2. Discretization of the equilibrium equation

Similarly, the equilibrium equation can be discretized as [64]

$$\mathbf{K}^u \mathbf{d} = \mathbf{f}^m + \mathbf{f}^{eq-\Theta} \quad (56)$$

where the extended nodal displacement vector $\mathbf{d}$ is defined

based on the standard nodal $\mathbf{d}^{st}$ and enriched $\mathbf{d}^{en}$ unknowns as

$$\mathbf{d} = \{ \mathbf{d}^{st} \quad \mathbf{d}^{en} \} \quad (57)$$

The stiffness matrix $\mathbf{K}^u$, the mechanical force vector $\mathbf{f}^m$, and the equivalent thermal force $\mathbf{f}^{eq-\Theta}$ are assembled from each node i :

$$\mathbf{K}_i^u = \int_{\Omega} (\mathbf{B}_i^u)^T \mathbf{D} (\mathbf{B}_i^u) d\Omega \quad (58)$$

$$\mathbf{f}_i^m = \int_{\Gamma_t} (\mathbf{N}_i^u)^T \bar{\mathbf{t}} d\Gamma + \int_{\Omega} (\mathbf{N}_i^u)^T \mathbf{b} d\Gamma \quad (59)$$

$$\mathbf{f}_i^{eq-\Theta} = \int_{\Omega} (\mathbf{B}_i^u)^T \mathbf{D} \varepsilon^\Theta d\Gamma \quad (60)$$

where $\mathbf{D}$ is the constitutive matrix and ε^Θ is the thermal strain, as defined in section 2. $\mathbf{N}_i^u$ and $\mathbf{B}_i^u$ are matrix of displacement shape functions and their derivatives, respectively, as

$$\mathbf{N}_i^\Theta =$$

for standard nodes :

$$\begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix}$$

for heaviside enriched nodes

$$\begin{bmatrix} N_i & 0 & N_i(H(\zeta) - H(\zeta_i)) & 0 \\ 0 & N_i & 0 & N_i(H(\zeta) - H(\zeta_i)) \end{bmatrix} \quad (61)$$

tip enriched nodes

$$\begin{bmatrix} N_i & 0 & N_i(F_1^u(\mathbf{x}) - F_1^u(\mathbf{x}_i)) & 0 \\ 0 & N_i & 0 & N_i(F_1^u(\mathbf{x}) - F_1^u(\mathbf{x}_i)) \end{bmatrix}$$

$$\mathbf{B}_i^u =$$

for standard nodes :

$$\begin{bmatrix} N_{i,x_1} & 0 \\ 0 & N_{i,x_2} \\ N_{i,x_2} & N_{i,x_1} \end{bmatrix}$$

for heaviside enriched nodes

$$\begin{bmatrix} N_{i,x_1} & 0 & (N_i(H(\zeta) - H(\zeta_i)))_{,x_1} & 0 \\ 0 & N_{i,x_2} & 0 & (N_i(H(\zeta) - H(\zeta_i)))_{,x_2} \\ N_{i,x_2} & N_{i,x_1} & (N_i(H(\zeta) - H(\zeta_i)))_{,x_2} & (N_i(H(\zeta) - H(\zeta_i)))_{,x_1} \end{bmatrix}$$

tip enriched nodes

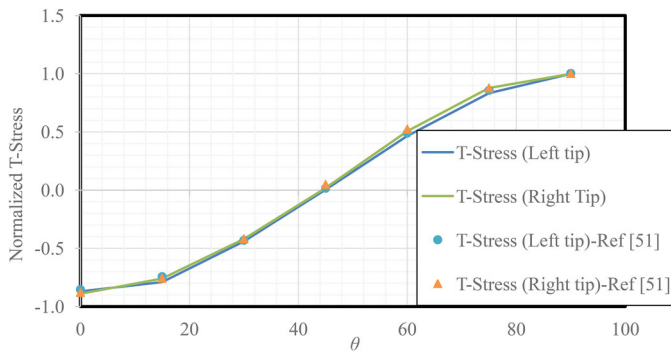
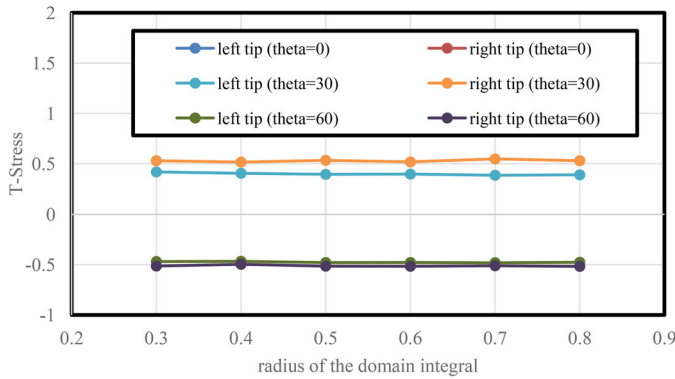
$$\begin{bmatrix} N_{i,x_1} & 0 & (N_i(F_1^u(\mathbf{x}) - F_1^u(\mathbf{x}_i)))_{,x_1} & 0 \\ 0 & N_{i,x_2} & 0 & (N_i(F_1^u(\mathbf{x}) - F_1^u(\mathbf{x}_i)))_{,x_2} \\ N_{i,x_2} & N_{i,x_1} & (N_i(F_1^u(\mathbf{x}) - F_1^u(\mathbf{x}_i)))_{,x_2} & (N_i(F_1^u(\mathbf{x}) - F_1^u(\mathbf{x}_i)))_{,x_1} \end{bmatrix} \quad (62)$$

7.4. Numerical integration

The Gauss integration rule is used for the standard elements. In Heaviside and tip enriched elements, the standard Gauss integration cannot provide sufficient accuracy, and

Table 1. Comparison of T-stress for different angles with the reference results.

		T-Stress							
βa	theta	Left crack tip				Right crack tip			
		Present XFEM	Ref. [51]	Ref. [52]	Ref. [8]	Present XFEM	Ref. [51]	Ref. [52]	Ref. [8]
0	0	-0.976	-0.947	-0.999	-0.983	-0.976	-0.947	-0.999	-0.983
	15	-0.847	-0.844	-0.866	-0.853	-0.847	-0.844	-0.866	-0.853
	30	-0.488	-0.488	-0.5	-0.497	-0.488	-0.488	-0.5	-0.497
	45	0.004	0.002	0	-0.005	0.004	0.002	0	-0.005
	60	0.495	0.5	0.5	0.491	0.495	0.5	0.5	0.491
	75	0.863	0.867	0.866	0.859	0.863	0.867	0.866	0.859
0.5	90	1.010	1.003	1	0.995	1.010	1.003	1	0.995
	0	-0.870	-0.854	-0.876	-0.858	-0.890	-0.879	-0.867	-0.896
	15	-0.790	-0.743	-0.763	-0.747	-0.760	-0.757	-0.763	-0.773
	30	-0.440	-0.431	-0.444	-0.436	-0.420	-0.418	-0.444	-0.434
	45	0.004	0.016	0.01	0.011	0.018	0.049	0.01	0.036
	60	0.466	0.49	0.49	0.484	0.507	0.525	0.49	0.513
	75	0.833	0.857	0.858	0.85	0.878	0.878	0.858	0.868
	90	0.999	1.003	1	0.994	0.999	1.003	1	0.994

**Figure 11.** Comparison of T-stress for different angles with the reference results ($\beta = 0.5$).**Figure 12.** Variations of the T-stress for angles 0° , 30° and 60° ($\beta a = 0.5$).

the sub-triangulation technique is used to compute all integrations, as described in Figure 7.

After sub-triangulation, the standard integration procedure is performed for each sub-triangle.

7.5. Solution procedure

The solution procedure for an adiabatic crack problem can be summarized into the following steps:

Step 1: Find the discontinuous temperature distribution by solving Eq. (48)

Step 2: Find the equivalent force resulted from the thermal strain by Eq. (60)

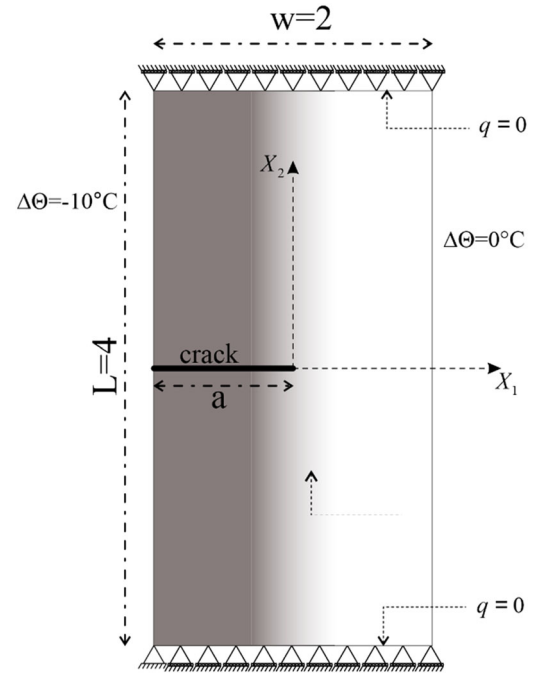
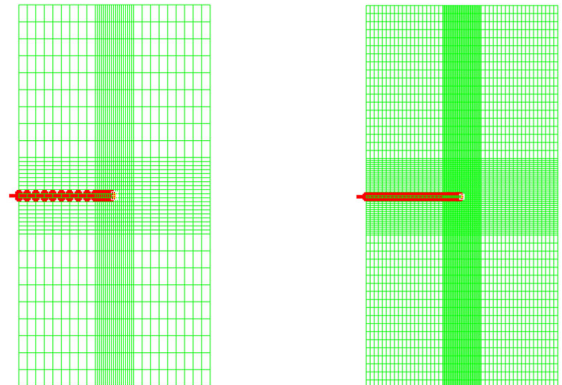
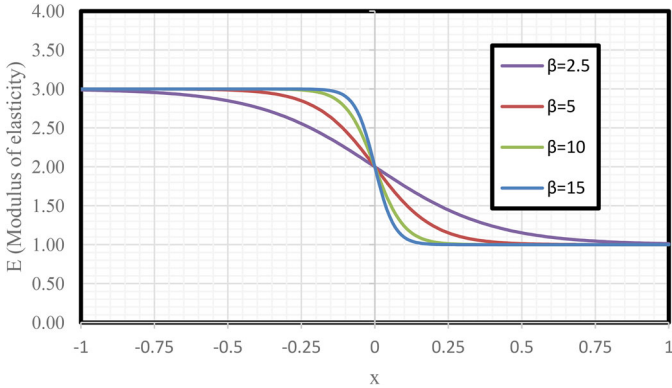
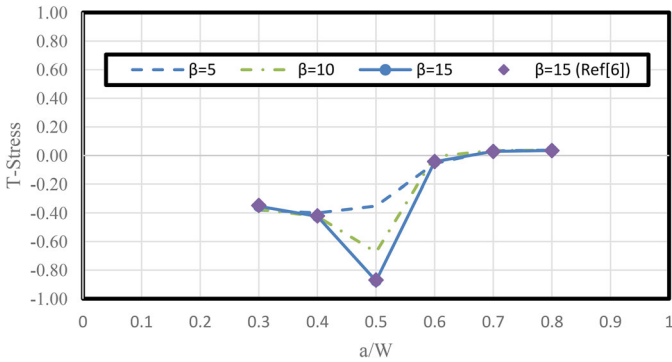
**Figure 13.** Geometry and boundary conditions of an edge cracked rectangular FGM plate.**Figure 14.** Mesh configurations of the edge cracked rectangular plate.

Table 2. Comparison of T-stress for present XFEM formulation and references ($\beta = 15, \delta = 5$) based on Mesh 1.

a/W	Present XFEM	Ref [6]	Differences (%)	Present XFEM (Without FGM term)
0.30	-0.3535	-0.3476	1.70	-0.3580
0.4	-0.4237	-0.4206	0.75	-0.2490
0.5	-0.8748	-0.8695	0.61	-0.1740
0.6	-0.0438	-0.0427	2.59	-0.0318
0.7	0.0284	0.0278	2.10	0.0276
0.8	0.0342	0.0339	0.74	0.0340

**Figure 15.** Variation of modulus of elasticity with respect to β .**Table 3.** Comparison of T-stress for two different meshes.

β	Mesh 1	Mesh 2	Difference (%)
5	0.351	0.349	0.57%
15	0.875	0.902	3%

**Figure 16.** Variations of T-stress versus crack length for $\beta = 5$, $\beta = 10$ and $\beta = 15$.

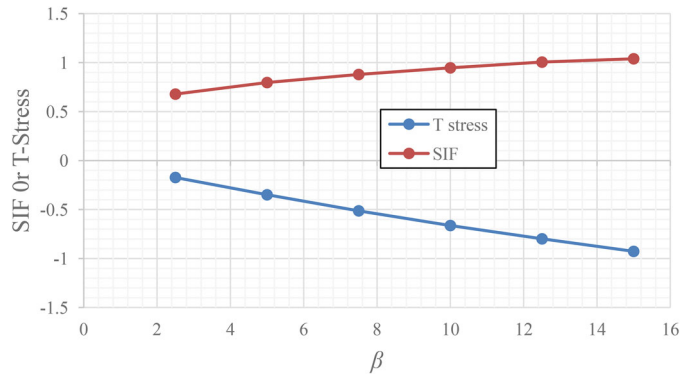
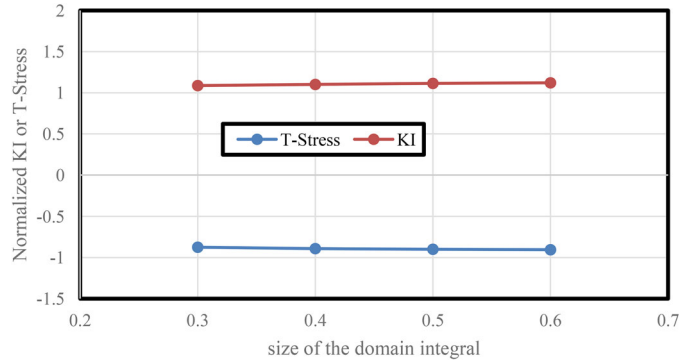
Step 3: Find the discontinuous displacement field by solving Eq. (56)

Step 4: Find SIFs:

- 4.1. Calculate the auxiliary displacement, stress and strain fields (Eqs. 15–17)
- 4.2 Find the M-integral parameters for each crack tip (Eq. 12)
- 4.3 Find the first and second modes of SIFs (Eq. 22)

Step 5: Find the T-stress:

- 5.1. Calculate the auxiliary displacement, stress and strain fields (Eqs. 19–21)
- 5.2 Find the M-integral parameters for each crack tip (Eq. (12))

**Figure 17.** Variation of SIF and T-stress in terms of non-homogeneity parameter β for $a/W = 0.5$.**Figure 18.** Variation of normalized K_I and T-stress for four different sizes of the domain integrals.

5.3 Find the T-stress (Eq. 25)

Step 6: Given a characteristic length r_c , find the crack propagation angle using Eq. (29)

Step 7: Check for crack propagation if the propagation criterion is satisfied (Eq. 31).

8. Numerical simulations

8.1. Example 1

To verify the developed formulation, a crack under thermo-mechanical loading is investigated. An inclined crack in the center of a rectangular FGM plate, studied in [6] by the finite element method, is considered. The geometry of plate is shown in Figure 8. The two sides of the plate are clamped and crack surfaces are assumed isothermal [6]. A total of 1,521 4-node elements are used in the modeling. Material parameters are considered unitless, as reported in [6]:

Geometric parameters :

$$a = 1, \quad L = W = 10, \quad \bar{\theta} = 0^\circ \sim 90^\circ$$

Mechanical properties :

$$E(X_1) = E^0 e^{\beta X_1}, \quad \nu(X_1) = \nu, \quad E^0 = 1, \quad \nu = 0.3$$

nonhomogeneity parameters :

$$\beta a = 0.0 \quad \text{and} \quad 0.5$$

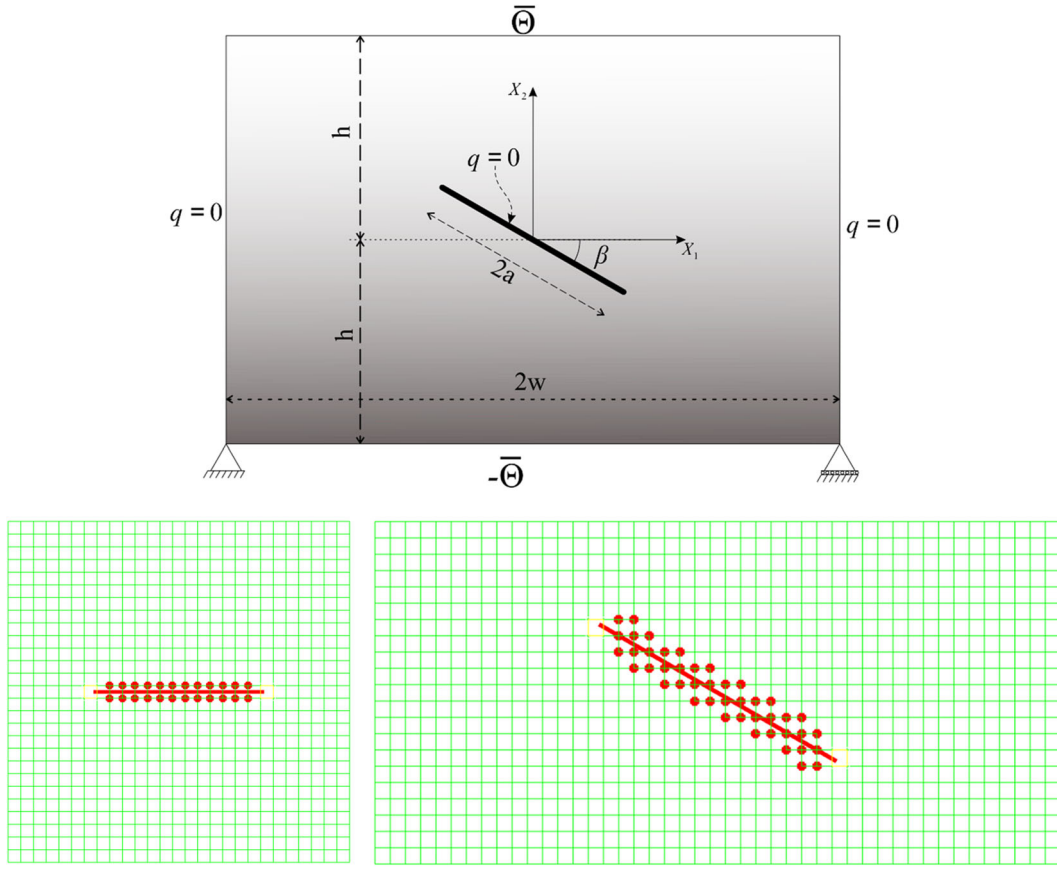


Figure 19. Geometry, boundary conditions and mesh configurations of the adiabatic crack problem.

The results of the present XFEM model in the case of $\beta = 0.5$ and for different angles of crack are compared with reference values in Figure 9 for K_I and K_{II} , respectively. SIFs in this figure are normalized with respect to $E^0 \sqrt{\pi a}$. All results are in acceptable ranges with the reference.

To investigate the path-independent nature of the M-integral method for nonhomogeneous materials, variations of the first and second modes of normalized stress intensity factors in six different size of the domain of integrals are shown in Figure 10 for angles of 0 and 60 degrees and in the case of $\beta = 0.5$.

Table 1 and Figure 11 present the T-stress in different crack angles. Closely a good agreement is observed with the reference results. Additionally, the path-independency of the adopted M-integral formulation to obtain the T-stress is investigated in Figure 12, which shows approximately the same results for six sizes of the domain integrals.

8.2. Example 2

In this problem, a horizontal edge crack in a rectangular plate under thermal gradient in parallel with the crack line is investigated. The region with high gradient of material properties is limited to a narrow band [6]. Geometry and boundary conditions are shown in Figure 13. The problem is solved with two different meshes, as depicted in Figure 14 (1,369 and 5,929 4-node elements for coarse and fine

meshes, respectively). Variation of material properties are defined as:

$$\begin{aligned} E(X_1) &= \frac{E^- + E^+}{2} + \frac{E^- - E^+}{2} \tanh(\beta X_1) \\ v(X_1) &= \frac{v^- + v^+}{2} + \frac{v^- - v^+}{2} \tanh(\delta X_1) \\ \alpha(X_1) &= \frac{\alpha^- + \alpha^+}{2} + \frac{\alpha^- - \alpha^+}{2} \tanh(\delta X_1) \\ k(X_1) &= \frac{k^- + k^+}{2} + \frac{k^- - k^+}{2} \tanh(\delta X_1) \end{aligned}$$

where the parameters are

$$\begin{aligned} \frac{a}{W} &= 0.3 - 0.8 \\ \beta &= 15 \quad \delta = 5 \\ (E^-, E^+) &= (1, 3), \quad (v^-, v^+) = (0.3, 0.1), \\ (\alpha^-, \alpha^+) &= (0.01, 0.03), \quad (k^-, k^+) = (1, 3) \end{aligned}$$

T-stress values for different ratios of a/W are presented in Table 2. Good agreement observed between the present XFEM model and the FEM reference results, where the maximum difference is limited to 2.6%, which is quite acceptable. Additionally, to investigate the effect of FGM terms in M-Integral formulation, the T-stress is calculated without the FGM terms in M-Integral formulation. The highlighted cell in Table 2 shows a significant error if FGM

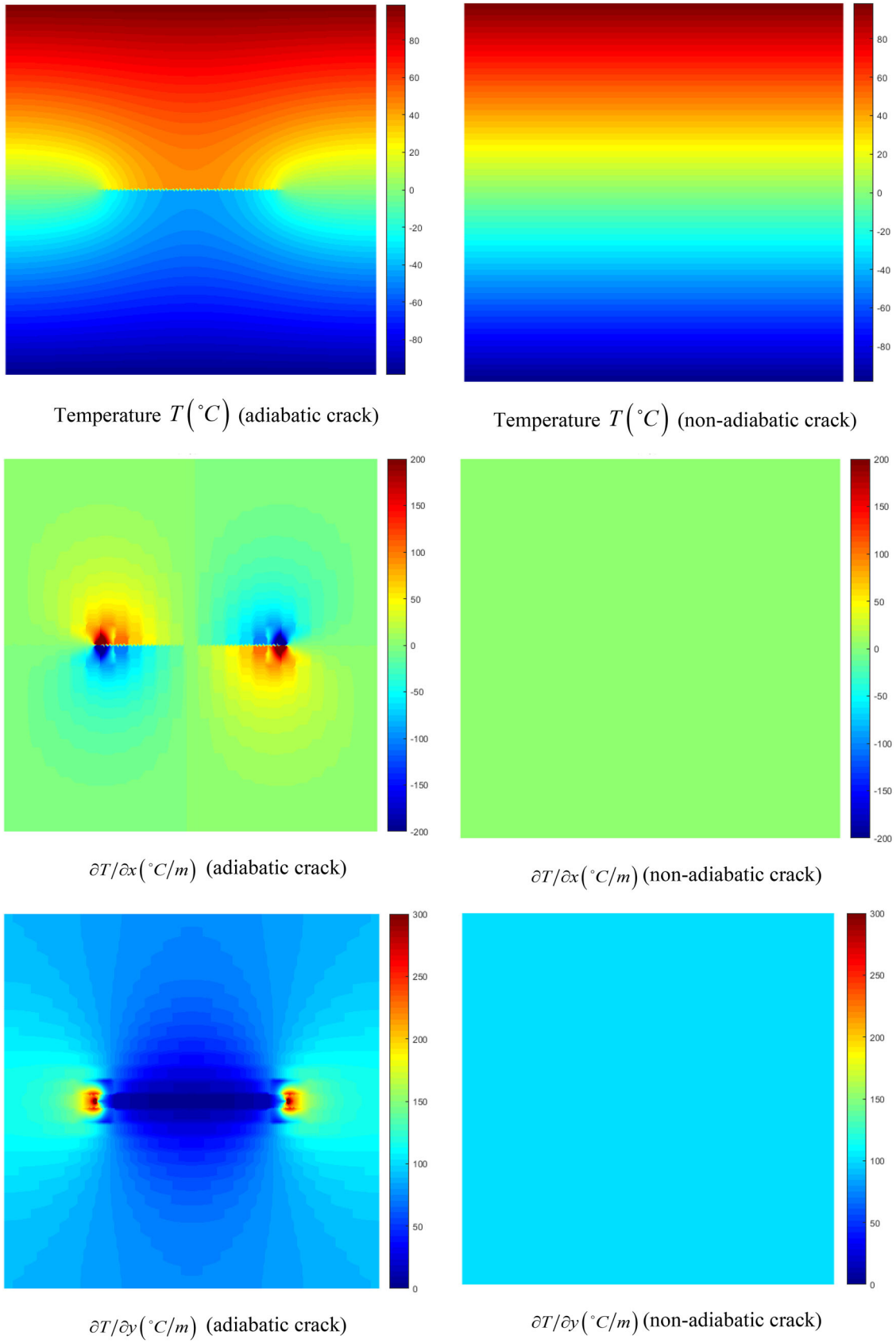


Figure 20. Comparison of the temperature field and its gradients in adiabatic and non-adiabatic cracks.

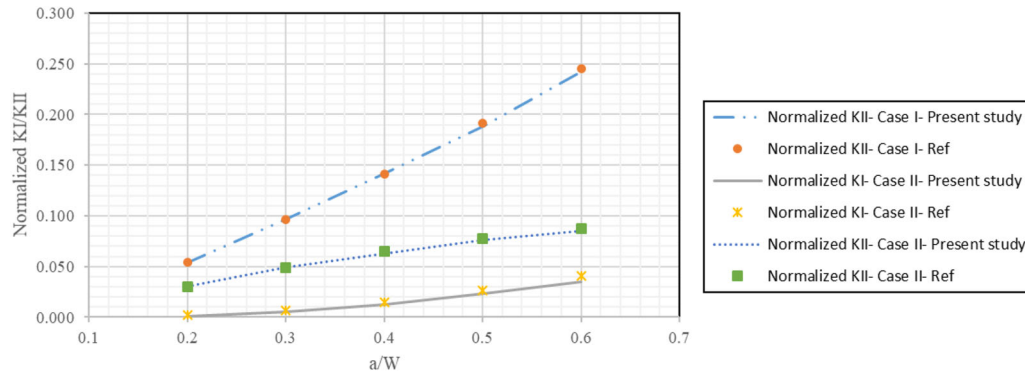


Figure 21. Comparison of normalized stress intensity factor in homogeneous cases I and II of the present study with reference [40].

Table 4. Material properties of ceramic/metal composite FGM [12].

Material	E (GPa)	ν	α ($^{\circ}\text{C}^{-1}$)	k (W/mK)
ZrO ₂	27.6	0.25	10.01×10^{-6}	1
Ni	175.8	0.25	13.91×10^{-6}	7

Table 5. Stress intensity factor, T-stress and initial crack propagation angle (with $r_c = 0.025$, $\beta = 30$, $W = 2L$, $T_0 = \alpha_{cr} E_{cr} \Theta$, $K_0 = \alpha_{cr} E_{cr} \sqrt{\pi a}$).

a/W	KI/K ₀		KII/K ₀		T-stress/T ₀		θ_p	
	Left	Right	Left	Right	Left	Right	Left	Right
0.2	0.058	0.050	0.176	-0.057	0.048	0.016	-66	57
0.3	0.067	0.078	0.251	-0.118	0.073	-0.083	-67	53
0.4	0.067	0.116	0.316	-0.176	0.138	-0.185	-69	50
0.5	0.052	0.164	0.362	-0.225	0.242	-0.261	-72	48
0.6	0.020	0.211	0.389	-0.264	0.356	-0.303	-73	47

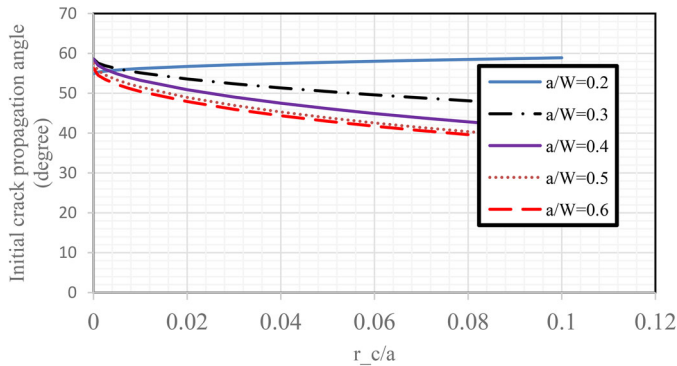


Figure 22. Variation of initial crack propagation angle in terms of r_c for the right tip.

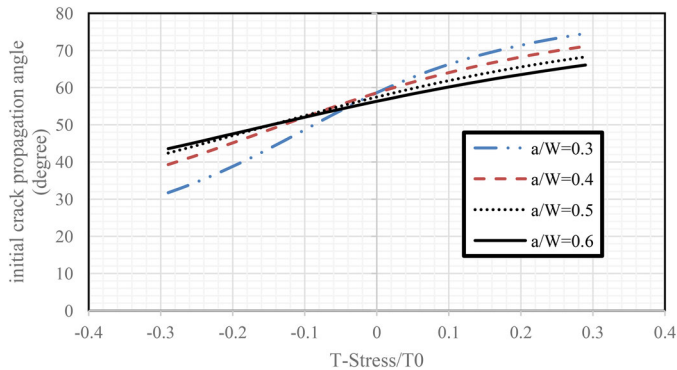


Figure 23. Variation of initial crack propagation angle in terms of the T-stress for the right tip ($T_0 = \alpha_{cr} E_{cr} \Theta$).

terms are omitted, which illustrates the considerable effect of additional terms in the FGM formulation of the M-integral. It should be noted that this considerable error (the highlighted cell) corresponds to $a/W = 0.5$ that falls in the FGM region, as depicted Figure 15, and the differences between the two cases (with and without FGM terms) decreases if the crack tip is located out of the FGM band (in the homogeneous region).

Values of T-stress for two meshes in case of $a/W = 0.5$ and $\beta = 5, 15$ are compared in Table 3. Differences in both cases are in the acceptable range. Nevertheless, in the case of $\beta = 15$, where severe gradients of mechanical properties exist, larger differences between the results of two meshes are observed.

To examine the effect of the size of FGM band in this problem, the T-stresses are calculated and compared for $\beta = 5$, $\beta = 10$ and $\beta = 15$. Variation of modulus of elasticity in two cases are depicted in Figure 15. The figure shows that the overall response approaches to nearly the bi-material response for larger β . Variations of the T-stress versus crack length are illustrated in Figure 16. The major difference at $a/W = 0.5$ can be attributed to the fact that $a/W = 0.5$ corresponds to the crack tip at the central FGM band, where a major difference in the gradient of modulus of elasticity is concentrated, as depicted in the Figure 15. By increasing the non-homogeneity parameter β , the plate approaches to a bi-material problem with sudden variation in mechanical properties. This situation can also be clearly observed from Figure 17, which shows variation of T-stress and SIF in terms of the non-homogeneity parameter β for $a/W = 0.5$. Clearly, the absolute values of both SIF and T-stress increase as the problem approaches to a bimaterial case. For example, SIF varies considerably from $K_I = 0.679$ ($\beta = 2.5$) to $K_I = 1.17$ ($\beta = 20$), and $|T| = 0.173$ ($\beta = 2.5$) to $|T| = 1.15$ ($\beta = 20$) that are considerable differences.

Furthermore, the path-independency of M-integral formulation in for both T-stress and K_I is clearly observed in Figure 18 based on four different sizes of domain integrals.

8.3. Example 3

In this example, a problem with adiabatic crack, where the heat flux normal to the crack faces is zero, is considered, as

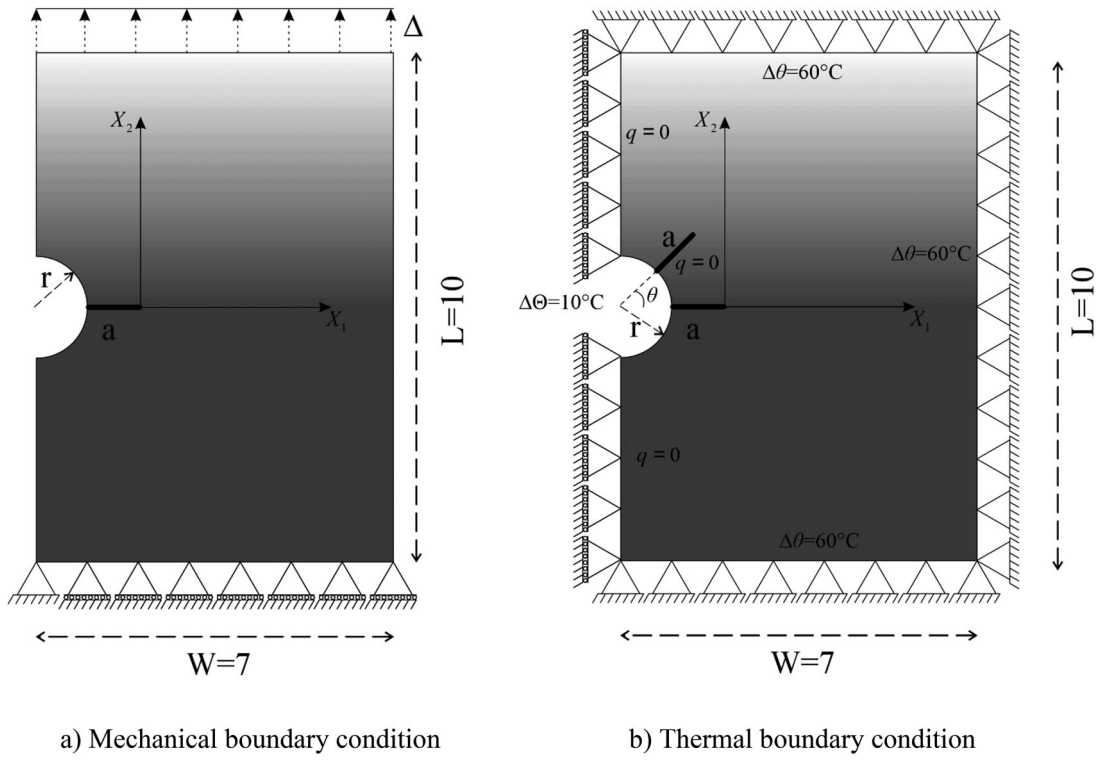


Figure 24. Mechanical and thermal boundary condition.

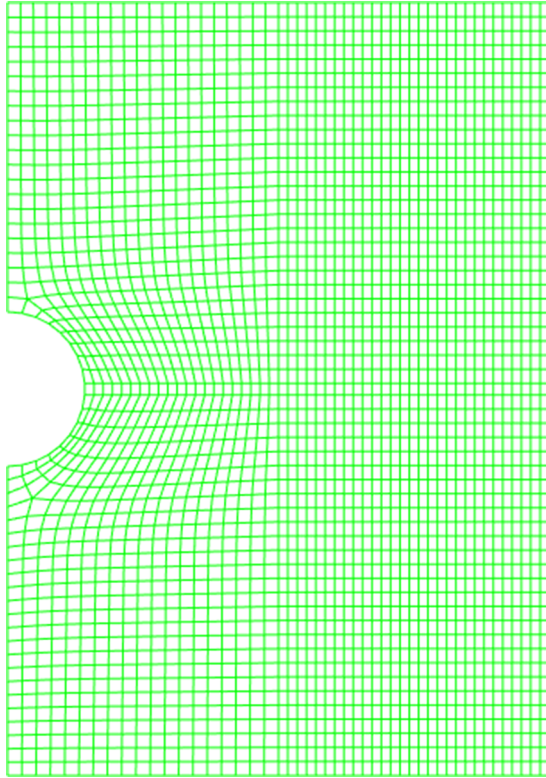


Figure 25. Mesh configuration of a plate with a crack emanating from the edge of semi-circular hole.

originally studied by [40] using XFEM (without the T-stress effects). Geometry, boundary conditions and mesh configuration are shown in Figure 19. Two cases of the pure second mode of fracture with $\beta = 0$ and $W = h$ (case I) and the mixed mode condition with $\beta = 30$ and $W = 2h$ (case II)

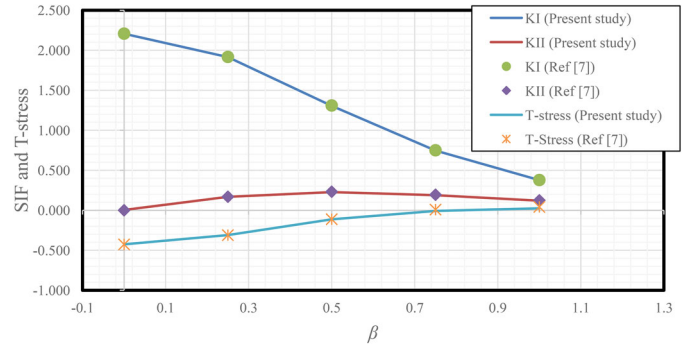


Figure 26. Comparison of the stress intensity factors and T-stress in the present study with reference [7] under mechanical loading and different non-homogeneity parameter β .

Table 6. Stress intensity factor and T-stress for a plate with a semi-circular hole.

$\beta = \gamma = \delta$	$\theta = 0\text{deg}$			$\theta = 45\text{deg}$		
	KI	KII	T	KI	KII	T
0.00	0.593	0.0	-0.564	0.539	-0.014	-0.599
0.25	0.557	-0.087	-0.547	0.725	-0.176	-1.202
0.50	0.493	-0.127	-0.509	0.956	-0.415	-2.486
0.75	0.466	-0.173	-0.464	1.326	-0.789	-4.905
1.00	0.487	-0.213	-0.399	1.916	-1.429	-9.980

are examined [40] with 729 and 945 4-node elements for cases I and II, respectively. Modulus of elasticity (E), Poisson's ratio (ν), thermal expansion coefficient (α) and thermal conductivity (k) are 174 (GPa), 0.3, 1.1×10^{-5} ($^{\circ}\text{C}^{-1}$) and 3 ($\text{W}/\text{m}^{\circ}\text{C}$), respectively, and the plane strain condition is considered.

First, the effect of adiabatic crack on stress and temperature field distributions in case I is considered. The boundary

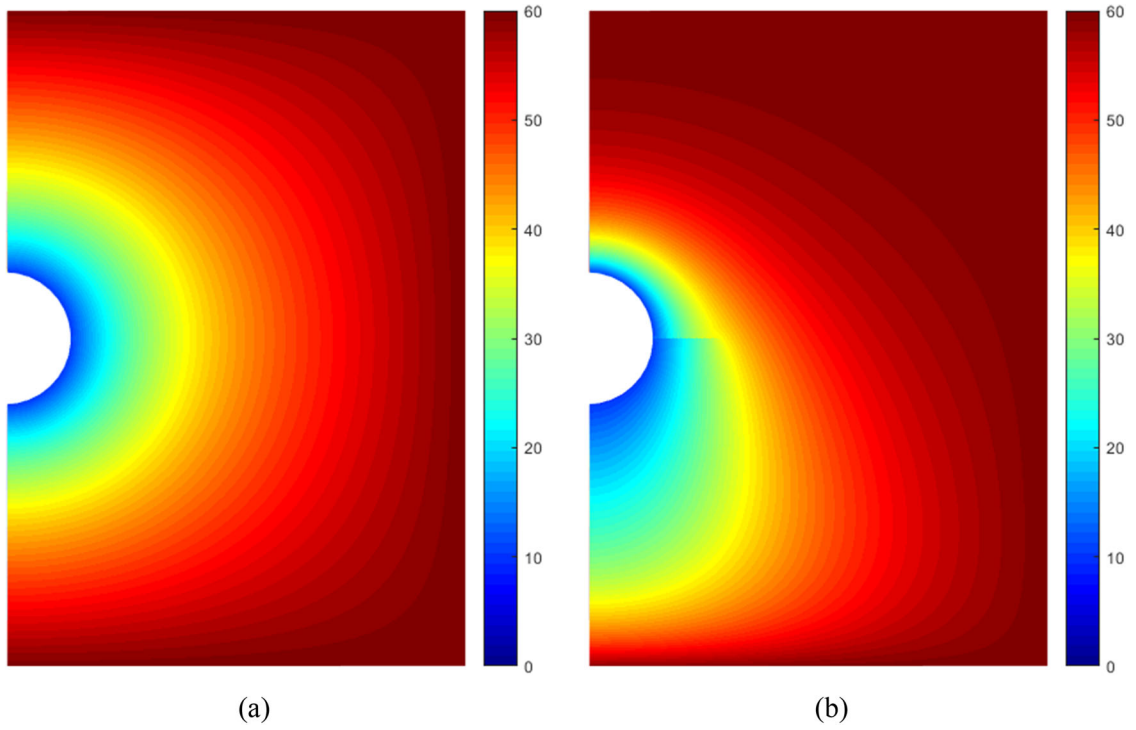


Figure 27. Temperature distribution in a) homogeneous $\delta = 0$ and b) non-homogeneous $\delta = 1$ cases.

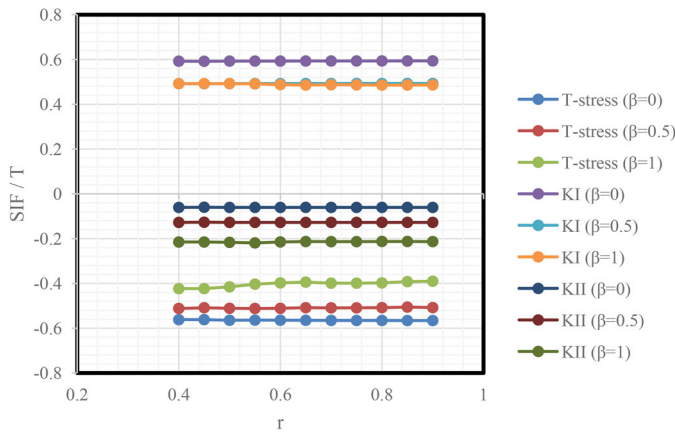


Figure 28. Path-independency of domain integrals for different fracture parameters ($\theta = 0^\circ$).

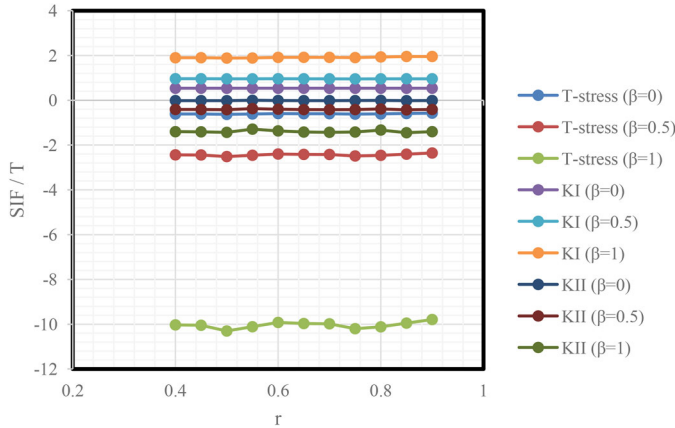


Figure 29. Path-independency of domain integrals for different fracture parameters ($\theta = 45^\circ$).

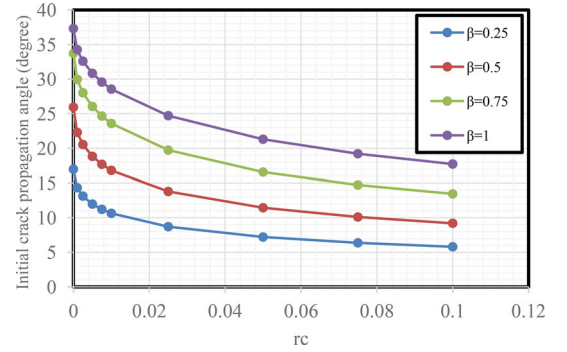


Figure 30. Initial crack propagation angle in terms of the characteristic length ($\theta = 0\text{deg}$).

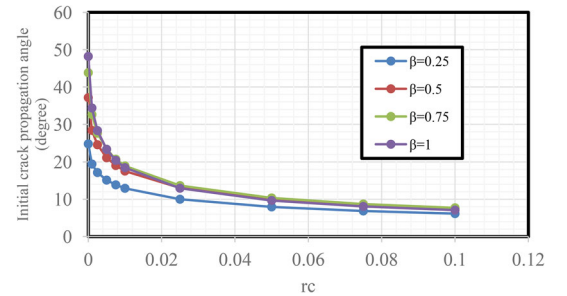


Figure 31. Initial crack propagation angle in terms of the characteristic length ($\theta = 45\text{deg}$).

conditions of the adiabatic crack are shown in Figure 19. First, comparison of the temperature field and its gradients for the two assumptions of adiabatic and non-adiabatic cracks are shown in Figure 20. Clearly, the adiabatic crack has a considerable effect in changing the temperature distribution and generates a singular heat flux around the crack

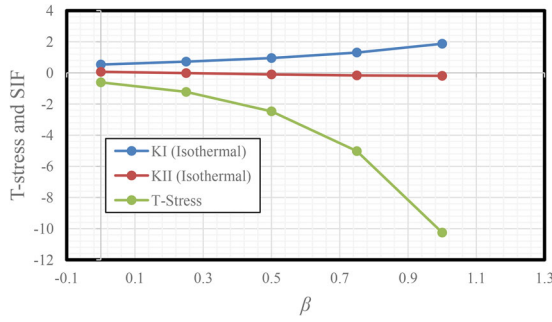


Figure 32. First and second mode of stress intensity factors, and the T-stress for isothermal condition.

tip, while in a non-adiabatic crack, the crack is practically ignored in solving the thermal equation.

The normalized SIF factor in the plane strain condition in the homogeneous case is compared with [40], as shown in Figure 21 for cases I and II. SIFs are normalized by $E\alpha\bar{\Theta}\sqrt{L}$. Note that case I represents the pure second mode of fracture; $K_I = 0$. Clearly, very good agreement is observed between the present results and ref [40].

Now, the material is assumed an FGM composed of a ceramic/metal composite (ZrO_2/Ni), where 100% ZrO_2 and Ni are assumed in top and bottom layers, respectively. Variation of material properties are defined as [12]:

$$\begin{aligned}
 E(y) &= E_{cr} + (E_m - E_{cr}) \left(\frac{h-y}{2h} \right)^{1.5} & -h < y < h \\
 \nu(y) &= \nu_{cr} + (\nu_m - \nu_{cr}) \left(\frac{h-y}{2h} \right)^{1.5} & -h < y < h \\
 \alpha(y) &= \alpha_{cr} + (\alpha_m - \alpha_{cr}) \left(\frac{h-y}{2h} \right)^2 & -h < y < h \\
 k(y) &= k_{cr} + (k_m - k_{cr}) \left(\frac{h-y}{2h} \right)^2 & -h < y < h
 \end{aligned}$$

and the parameters are presented in Table 4.

Geometry of this case is depicted in Figure 19 with $\beta = 30$ and $W = 2L$ (the same as case II), and the plane stress condition is considered. Normalized stress intensity factors and T-stress are reported in Table 5. Generally, the second mode of fracture is dominant in this type of loading and boundary condition, and K_{II} increases by increasing a/W . The same behavior is observed in T-stress. However, in the right tip, a change in the sign of T-stress affects the crack propagation angle.

One of the important effects of T-stress is on the initial crack propagation angle. The role of T-stress on crack propagation angle not only depends on its values but also on the characteristic length r_c . For example, the initial crack propagation angle in the case of $r_c = 0.025$ is presented in Table 5. Figure 22 shows the variation of initial crack propagation angle θ_p in terms of r_c for the right tip. It further shows that except for $a/W = 0.2$, the crack propagation angle decreases by increasing the characteristic length r_c . The inverse behavior of $a/W = 0.2$ can be attributed to the positive sign of T-stress in this case. To better consider the effect of T-stress on the crack propagation angle, variations of θ_p in terms of the T-stress are investigated in Figure 23, for the case of $r_c = 0.05$ in the right tip. Figure 23 shows that, in comparison to the case of $T = 0$, θ_p increases when $T > 0$ and vice versa. Moreover, Figure 22 and Figure 23 show that the effect of T-stress on the deviation of crack propagation angle is increased

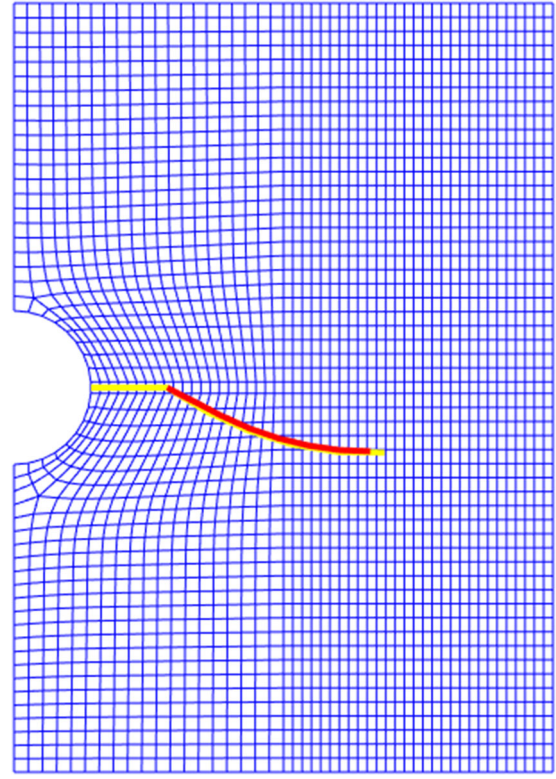


Figure 33. Crack propagation under the thermal loading with $\beta = 0.5$ (Yellow path: increment $da = 0.6$, Red path: increment $da = 0.4$).

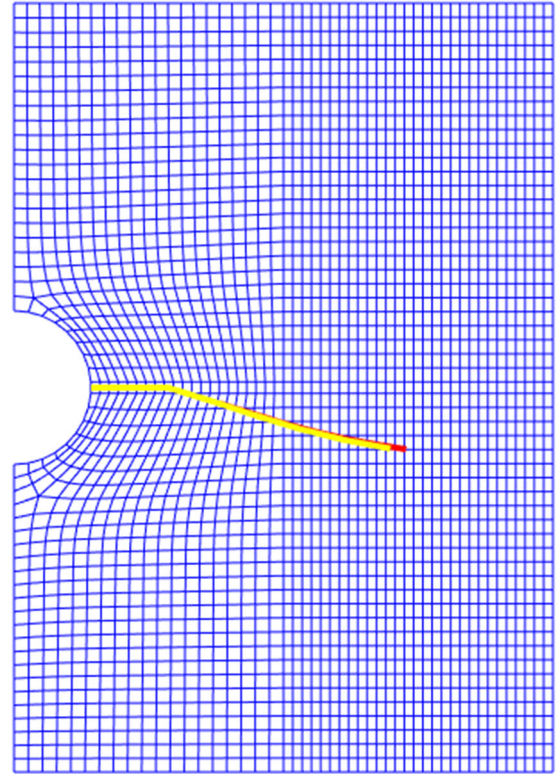


Figure 34. Crack propagation under mechanical loading with $\beta = 0.5$ (Yellow path: increment $da = 0.6$, Red path: increment $da = 0.4$).

when the parameter $T\sqrt{r_c}$ increases. This finding confirms the effect of the third term in Eq. (29) in comparison with the cases where the T-stress is neglected.

8.4. Example 4

In this example, an edge crack emanating from a circular hole in an FGM plate is investigated. This problem was originally presented in [7] and analyzed using FEM under mechanical loading with boundary conditions presented in

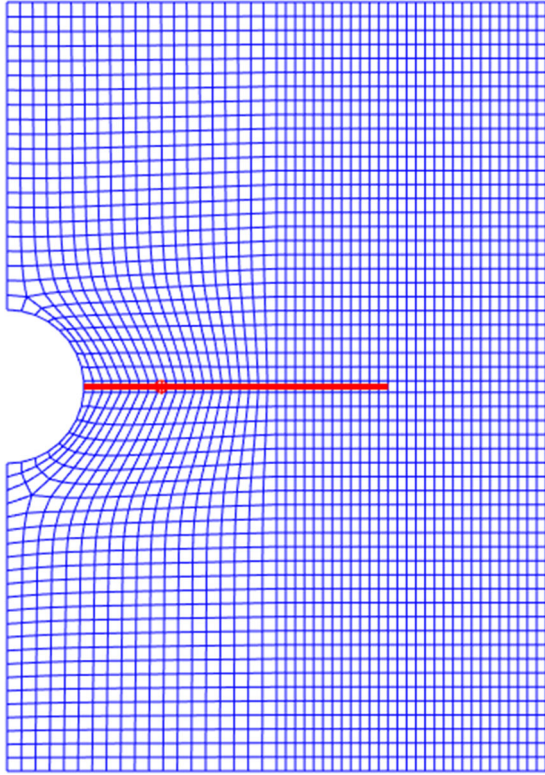


Figure 35. Crack propagation under thermal loading for homogeneous problem with $\beta = 0$ (pure mode I).

Figure 24-a. The modulus of elasticity is considered as $E = \bar{E}e^{\beta x_2}$ where $\bar{E} = 1$ and β is in the range of $0 - 0.75$, with a constant Poisson's ratio of $\nu = 0.3$.

This problem is now solved under a displacement loading $\Delta = 10$ on the upper edge similar to [7], as presented in Figure 24-a. A mesh of 2,714 4-node elements is used, as depicted in Figure 25.

The stress intensity factors and T-stress obtained by the present approach are compared with ref [7] under mechanical loading in Figure 26 for different values of βa , which shows a very good agreement.

Now, in order to solve the problem for the case of adiabatic crack under thermal loading, the boundary conditions depicted in Figure 24-b are considered. To apply the adiabatic crack conditions, the heat flux ($q = k(\nabla T)$) is set to zero on crack surfaces. The thermal expansion and conductivity properties are assumed to vary by the exponential functions $\alpha = \bar{\alpha}e^{\gamma y}$ and $k = \bar{k}e^{\delta y}$ with $\bar{\alpha} = 0.01$ and $\bar{k} = 1$. The power factors γ and δ are assumed to be equal to β (in modulus of elasticity). The initial uniform temperature is set to 60°C .

Normalized SIFs and T-stress are reported in Table 6 for straight and inclined cracks, with different non-homogeneous parameters (β, γ, δ). Moreover, contours of temperature in homogeneous ($\delta = 0$) and non-homogeneous ($\delta = 1$) cases are compared in Figure 27. The figure clearly shows that the symmetric condition is violated in the FGM condition and the effect of crack should be considered in the thermal equation.

Path-independency of domain integrals is clearly observed for SIFs and T-stress in cases of $\theta = 0^\circ$ and $\theta = 45^\circ$ in Figures 28 and 29, respectively.

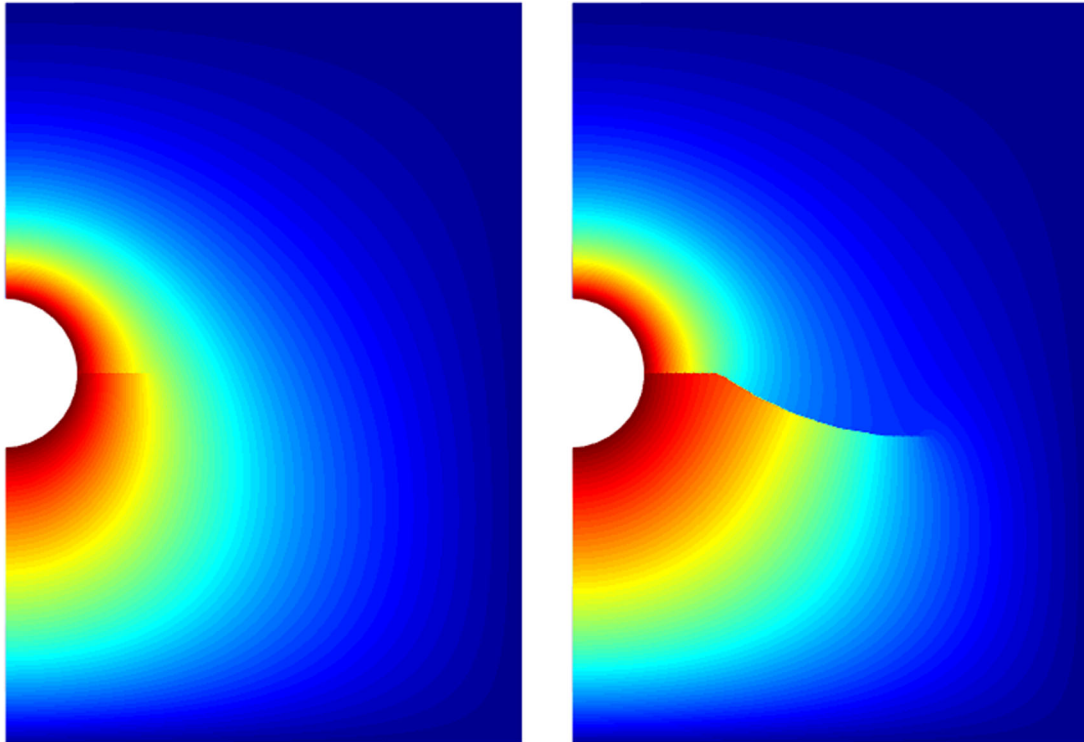


Figure 36. Temperature contour in the first step (left) and last step (right) under thermal loading with $\beta = 0.5$.

Figures 30 and 31 illustrate the initial crack propagation angles in terms of the characteristic length r_c . These figures also show that the deviation of crack angle with respect to the case of $r_c = 0$ approaches a constant value by increasing r_c .

Additionally, Figure 32 shows variations of different fracture parameters (KI, KII and T-stress) in terms of the non-homogeneity parameter β with $\theta = 45^\circ$ for non-adiabatic model that heat flux is continuous across the crack surfaces.

Finally, crack propagation under thermal and mechanical loading is considered in this part. Figures 33 and 34 show the crack propagation paths under thermal and mechanical loadings, respectively, for $\beta = 0.5$. In both cases, the crack incremental lengths $da = 0.4$ and $da = 0.6$ are compared. The identical paths for the two increments clearly show the insensitivity of the response.

Additionally, the crack propagation path in case of $\beta = 0$ is examined in Figure 35 under the thermal loading. This figure predicts a straight crack path, which is related to pure fracture mode I when the non-homogeneity parameter β is zero (and the material is homogeneous). It is clear that in cases of $\beta \neq 0$, a mixed mode condition governs the problem, and the crack deviates from its initial path (as presented in Figures 33 and 34).

The temperature distributions under the thermal loading with $\beta = 0.5$ in the first and last increments are presented in Figure 36.

9. Conclusion

This study was dedicated to the investigation of T-stress of adiabatic cracks in FGMs under the thermo-mechanical loadings based on the interaction integral method. The effect of T-stress on crack initiation angle was studied and the crack propagation under the thermal and mechanical loadings was examined using XFEM. Since the insulated surfaces of adiabatic cracks lead to zero-temperature flux across crack surfaces, a model of singular temperature gradient/flux around the crack tips was considered in XFEM formulation by appropriate crack tip enrichments for the temperature and displacement fields.

Path-independent results of T-stress for adiabatic cracks based on the robust J-integral formulation were explored in different examples. Additionally, it was investigated that the effect of incompatible terms of the interaction integral formulation on the result of T-stress should be included in higher gradient of material properties. Furthermore, it was numerically shown that the absolute values of SIFs and T-stress would considerably increase when the smooth FGM materials approached to a layered bimaterial; approving the superiority of FGMs with respect to layered composites.

Finally, it was concluded that the initial crack propagation angle would deviate from the standard propagation angle (without T-stress term) by increasing the size of the fracture process zone (characteristic crack length). Crack propagation paths under thermo-mechanical loading using XFEM formulation would remain insensitive to crack incremental length, regardless of the mesh configuration.

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