III European Conference on Computational Mechanics
Solids, Structures and Coupled Problems in Engineering
C.A. Mota Soares et.al. (eds.)
Lisbon, Portugal, 5–8 June 2006

A NEW APPROACH FOR ELIMINATION OF DISSIPATION AND DISPERSION ERRORS IN PARTICLE METHODS

Hassan Ostad-Hossein[1], and Soheil Mohammadi[2]

[1]School of Civil Engineering, University of Tehran, Tehran, Iran
(hostad@ut.ac.ir).

[2]School of Civil Engineering, University of Tehran, Tehran, Iran
(smoham@ut.ac.ir).

Keywords: Dispersion Error, Dissipation Error, Truncation Error, Particle Methods

Abstract. *Numerical errors may be introduced in some numerical methods of solving differential equations because of their nature. Discretizing a continuum medium would result in changing the wave velocity and inducing numerical errors into the solution. Some methods using strong formulations are based on the Taylor expansion. Therefore, using only a finite number of Taylor series terms for particle simulations introduces truncation errors.*

Truncation of the Taylor expansion is also the reason for developing two other types of error. The first, called dispersion error appears in the form of extra vibration in high frequency modes that can result in solution instability in some problems. Another type of error is dissipation and may cause decrease in wave amplitude.

Particle methods such as SPH [1] and CSPM [2], are also involved with truncation errors. A number of methods have already been proposed for removing dispersion from particle methods such as adding artificial stress. However these methods become energy dissipative resulting in wave amplitude decays after several time steps.

In this paper further investigation is performed to study the roots of dispersion and dissipation errors in particle methods. A new procedure is proposed for eliminating dispersion and stabilizing the solution, based on the CSPM particle method and the Newmark time integration scheme. The results are compared with other existing methods.

1 INTRODUCTION

The smoothed particle hydrodynamics method, SPH, was originally developed by Gingold, Monaghan and independently by Lucy in astrophysical problems. The method has been widely applied in many fields such as hydrodynamics, solid mechanics and simulating many other natural phenomena. Nevertheless, it has encountered some physical and mathematical problems, such as lack of consistency in boundaries and failure in satisfying boundary conditions. Chen et al [1,2] proposed a new method by using the Taylor expansion series which can satisfy consistency conditions needed for the second order problems, and boundary conditions can be directly applied.

Another problem associated with SPH simulations in some applications is its instability, and loss of accuracy. Particularly in modeling solid media, this problem may cause tensile instability resulting in particle clumping similar to phenomenon of creation of non-physical fracture in brittle materials that may be difficult to distinguish physical fracture from nonphysical one [5].

Many methods have been already proposed to remove this instability. It was first studied by Swegle et al. who related it to the signs of pressure and the second derivative of the interpolating kernel. A short-wavelength instability identical to tensile instability was analyzed by Philips and Monaghan for the case of magneto gas dynamics. Many efforts have been done over the years to modify interpolation kernel for removing tensile instability. Randles and Libersky used dissipation terms to remove the instability, while Monaghan et al, presented artificial stress approach which improved the results for fluid and solid dynamics problems [5].

The most conventional method is adding an artificial term to the stress state in equilibrium equation in hydrodynamics or solid mechanics problems.

In this paper, first the main source of numerical error is investigated and then methods of prevention are discussed. The method of artificial stress will be reviewed and a new method using CSPM procedure without any artificial stress will be presented for solid mechanics problem. Numerical tests are used to compare the proposed approach with other available methods.

2 PARTICLE METHODS

2.1 SPH methodology

The SPH method offers a way of solving a differential equation in a strong form similar to the finite difference method. Superiority of SPH to the finite difference method is its capability of simulation of any medium with complex geometry and irregularly distributed set of particles. In addition, particles can move freely in a medium, which simplifies simulation of fluid dynamics and large deformation of solids in Lagrangian space.

SPH method is based on evaluation of values of a function and its derivatives at a particle in terms of values of the function and its derivatives in neighbor particles. The main concept of SPH is based on the following Dirac delta function property:

$$\int_{-\infty}^{+\infty} u(\xi)\delta(x_i-\xi)\,d\xi = u(x_i)\int_{-\infty}^{+\infty}\delta(x_i-\xi)\,d\xi = u(x_i) \tag{1}$$

where δ is the Dirac function, $u(x_i)$ is the value of function u at point x_i. SPH uses a weight kernel function w instead of the Dirac function, while assuming the main characteristics of the Dirac function. Therefore, equation (1) is transformed into:

$$u(x_i) \approx u^h(x_i) = \int_{\Omega} u(\xi) w(x_i - \xi)\, d\xi \qquad (2)$$

where Ω is the smoothing domain having a radius of two times of the smoothing length h. The discrete form of equation (2) becomes similar to a weighted average of the neighbor nodal values within the smoothing domain,

$$u^h(x_i) \approx \sum_{J=1}^{N} u_j w(x_i - x_j) \Delta v_j \qquad (3)$$

where N is the number of neighbor particles located in the smoothing domain Ω, $w(x_i - x_j)$ is the value of weight function that depends on the distance between nodes i and j and the smoothing length h, and Δv_j is the space that occupied by particle j. In one dimensional problems, Δv_j changes to Δx_j, that is an average distance from adjacent particles. By increasing the distance of nodes i, j, the effect of particle j in evaluation of a function value at particle i is reduced.

The first and second derivatives of a function can be estimated using equation (2) and the rule of integration by part. By choosing an appropriate weight function, residual or boundary terms of the integration by part can be neglected; therefore:

$$\left(\frac{\partial u}{\partial x_\alpha}\right)_i \approx \left(\frac{\partial u}{\partial x_\alpha}\right)_i^h = -\sum_{J=1}^{N} u_j \left(\frac{\partial w}{\partial x_\alpha}\right)_{ij} \Delta v_j \qquad (4)$$

$$\left(\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta}\right)_i \approx \left(\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta}\right)_i^h = \sum_{J=1}^{N} u_j \left(\frac{\partial^2 w}{\partial x_\alpha \partial x_\beta}\right)_{ij} \Delta v_j \qquad (5)$$

To remove residual terms of the integration by part in equation (4), the first derivative of the weight function should be anti symmetric, whereas in equation (5), it is difficult to remove all boundary terms by choosing an appropriate weight function.

Because of the lack of consistency of the SPH method in and near boundaries, many studies have been directed towards improving this property resulting in development of many modified SPH approaches; RKPM [6], CSPM [1,2] and MSPH [7].

2.2 CSPM methodology

CSPM was proposed by Chen et al. in 1999 [1,2] using the Taylor series. The Taylor expansion can be written as:

$$u(x) = u(x_i) + \left(\frac{\partial u}{\partial x}\right)_i .(x - x_i) + \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i .(x - x_i)^2 + O(\Delta x^3) \qquad (6)$$

For evaluation of the first derivative of u, the first two terms of the series are taken in to account. Multiplying (6) by a weight function w^a and neglecting higher order terms and integrating over the smoothing domain, results in equation (7) in a discretized form

$$(\frac{\partial u}{\partial x})_i^h = \frac{\sum_{j=1}^{N}(u(x_j)-u(x_i)).w_{ij}^a \Delta v_j}{\sum_{j=1}^{N} w_{ij}^a.(x_j - x_i)\Delta v_j} \tag{7}$$

In order to avoid zero denominator, the weight function w^a should be anti symmetric. Equation (7) ensures elimination of boundary errors that appear in the standard SPH .

For evaluation of the second derivative of u , the first three terms of the series are considered. Multiplying (6) by a symmetric weight function w^s and neglecting the fourth and higher order terms and integrating over the smoothing domain, results in one equation with two unknown variables $(\frac{\partial u}{\partial x})_i^h, (\frac{\partial^2 u}{\partial x^2})_i^h$. Variable $(\frac{\partial u}{\partial x})_i^h$ has already been calculated by (7), therefore, only $(\frac{\partial^2 u}{\partial x^2})_i^h$ remains to be obtained:

$$(\frac{\partial^2 u}{\partial x^2})_i^h \approx \frac{\sum_{j=1}^{N}(u(x_j)-u(x_i)).w_{ij}^s.\Delta v_j - (\frac{\partial u}{\partial x})_i^h.\sum_{j=1}^{N}(x_j - x_i)w_{ij}^s.\Delta v_j}{1/2\sum_{i=1}^{N}(x_j - x_i)^2 w_{ij}^s.\Delta v_j} \tag{8}$$

Derivative of w^a, or any other symmetric function may be chosen to replace w^s , in order to avoid zero denomator.

Another similar method called MSPH solves the first and second equations simultaneously [7]. Multi dimensional problems can be also solved by using multi dimensional Taylor expansion [1,2,7].

3 TRUNCATION ERROR

Differential equations can be solved in a strong form, using the Taylor expansion for determination of the values of derivatives of a function at a node in terms of its neighbor nodes.

Neglecting higher order derivatives in the Taylor series, results in approximate estimation of function derivatives, causing creation of truncation error. For example the first derivative of a time dependent function u is obtained by:

$$\begin{aligned} u_{i+1}^n &= u_i^n + \Delta x(\frac{\partial u}{\partial x})_i^n + \frac{(\Delta x)^2}{2}(\frac{\partial^2 u}{\partial x^2})_i^n + \frac{(\Delta x)^3}{6}(\frac{\partial^3 u}{\partial x^3})_i^n + O(\Delta x^4) \\ (\frac{\partial u}{\partial x})_i^n &= \frac{u_{i+1}^n - u_i^n}{\Delta x} - \frac{(\Delta x)}{2}(\frac{\partial^2 u}{\partial x^2})_i^n - \frac{(\Delta x)^2}{6}(\frac{\partial^3 u}{\partial x^3})_i^n - O(\Delta x^4) \end{aligned} \tag{9}$$

where n is the time step number and i is the particle number.

The error term in estimating the first derivative using the finite difference method can be defined as

$$(\frac{\partial u}{\partial x})_i^n = \frac{u_{i+1}^n - u_i^n}{\Delta x} + E_1(\Delta x)$$
$$E_1(\Delta x) = -\sum_{p=2}^{\infty} \frac{(\Delta x)^{p-1}}{p!} (\frac{\partial^p u}{\partial x^p})_i^n \tag{10}$$

The same equation can be written in time domain,

$$(\frac{\partial u}{\partial t})_i^n = \frac{u_i^{n+1} - u_i^n}{\Delta t} + E_1(\Delta t)$$
$$E_1(\Delta t) = -\sum_{p=2}^{\infty} \frac{(\Delta t)^{p-1}}{p!} (\frac{\partial^p u}{\partial t^p})_i^n \tag{11}$$

The procedure can lead to evaluation of the error for the second derivatives,

$$(\frac{\partial^2 u}{\partial x^2})_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + E_2(\Delta x)$$
$$E_2(\Delta x) = -\sum_{p=2}^{\infty} \frac{2.\Delta x^{2p-2}}{(2p)!} .(\frac{\partial^{2p} u}{\partial x^{2p}})_i^n \tag{12}$$

The same equation can then be written in time domain

$$(\frac{\partial^2 u}{\partial t^2})_i^n = \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} + E_2(\Delta t)$$
$$E_2(\Delta t) = -\sum_{p=2}^{\infty} \frac{2.\Delta t^{2p-2}}{(2p)!} .(\frac{\partial^{2p} u}{\partial t^{2p}})_i^n \tag{13}$$

Equations (10-13) define the total error in various PDEs with the maximum order of two. For instance, error of solving the first order partial differential equation using FDM can be obtained as,

$$\frac{\partial u}{\partial t} = c_0 \frac{\partial u}{\partial x}$$
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = c_0 \frac{u_{i+1}^n - u_i^n}{\Delta x} - E_1(\Delta t) + c_0.E_1(\Delta x) \tag{14}$$

$E_1(\Delta x)$, $E_1(\Delta t)$ can be calculated using (10), (11). By using closed form solutions of PDE in (14), time derivatives in equation (11) can be written in terms of displacement derivatives. Therefore, the total error E_{total} in numerical form of PDE in (14) can be written as,

$$E_{total} = -E_1(\Delta t) + c_0.E_1(\Delta x) = \sum_{p=2}^{\infty} \frac{c_0.(\Delta x)^{p-1}.((c_0.\frac{\Delta t}{\Delta x})^{p-1} - 1)}{p!} (\frac{\partial^p u}{\partial x^p})_i^n \tag{15}$$

It has been discussed in literature [8] that neglecting terms with odd order of derivatives in (15) will induce dispersion in solution, while neglecting even order derivatives will cause dissipation. As a result, the truncation error includes both dispersion and dissipation errors. Dissipation error appears in the form of amplitude decay in results, causing undesirable numerical damping. Dispersion error causes exciting vibration modes of short wave length and may finally cause instability of the solution, as depicted in Figure 1.

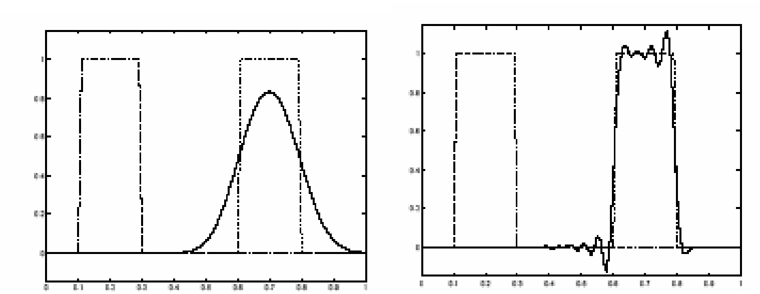

Figure 1: dissipation (left) and dispersion (right) errors

Another form of PDE that should be discussed here is the second order partial differential equation. Closed and FDM forms of this type are defined as:

$$\frac{\partial^2 u}{\partial t^2} = c_0^2 \frac{\partial^2 u}{\partial x^2}$$
$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} = c_0^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} - E_2(\Delta t) + c_0^2 . E_2(\Delta x) \tag{16}$$

The error term can be evaluated similar to the first order PDE:

$$E_{total} = -E_2(\Delta t) + c_0^2 . E_2(\Delta x) = \sum_{p=2}^{\infty} \frac{2.c_0^2 . \Delta x^{2p-2} . ((\frac{c_0 . \Delta t}{\Delta x})^{2p-2} - 1)}{(2p)!} . (\frac{\partial^{2p} u}{\partial x^{2p}})_i^n \tag{17}$$

Equation (17) implies that only even order derivatives appear in error. As a result, only dissipation errors are induced in results.
Adding terms similar to those of error in equations (15) or (17) to the P.D.E. is expected to reduce the truncation error.

Considering the fact that SPH and CSPM, similar to FDM are based on the Taylor series, and in some special cases, the CSPM with some simplification reduces to FDM formulation [3], truncation error is expected to similarly affect the results in all these particles methods.

4 REDUCTION OF 1-D WAVE PROPAGATION SOLUTION ERROR

4.1 Stress based procedure

In this method, the following three equations are written in strong forms at time step n:

Equilibrium equation:

$$(\frac{dv}{dt})^n = \frac{1}{\rho}(\frac{\partial\sigma}{\partial x})^n \qquad (18)$$

Compatibility equation:

$$\dot{\varepsilon}^n = (\frac{\partial\dot{u}}{\partial x})^n \qquad (19)$$

Constitutive relation:

$$\dot{\sigma}^n = E.\dot{\varepsilon}^n \qquad (20)$$

Equation (18) is identical to equation (14), therefore, using CSPM or similar derivative estimations based on the Taylor expansion leads to truncation errors, such as dissipation and dispersion.

By adding stabilizing terms to the equilibrium equation (18), these errors are prevented and more stable and more accurate solution is expected to be achieved. Since the equilibrium equation is identical to PDE in (14) these added terms are basically similar to error terms evaluated in (15).

Adding stabilizing terms have been discussed in several studies. One of the most conventional stabilizers was first proposed by Monaghan and Gingold (1983) and can be also used in the standard SPH and CSPM [1,2,3]:

$$\Pi_{ij} = \begin{cases} \dfrac{-\alpha.\bar{c}_{ij}.\mu_{ij} + \beta.\mu_{ij}^2}{\bar{\rho}_{ij}} & if\ v_{ij}.x_{ij} < 0 \\ 0 & otherwise \end{cases}$$

$$\mu_{ij} = \frac{h_{ij}.v_{ij}.x_{ij}}{\left|r_{ij}\right|^2 + \kappa h_{ij}{}^2} \qquad (21)$$

where $\bar{c}_{ij}$ is the average of sound velocity of particles i, j, $\bar{\rho}_{ij}$ is the average density of particles i, j. v_{ij} is the relative velocity of particles i, j and x_{ij} is the relative distance of particles i, j. α, β, κ are constants, and h_{ij} determines the smoothing length.

This stabilizing term is added to the stress term in equilibrium equation, Therefore the SPH form of the equilibrium equation can be written as

$$(\frac{dv}{dt})_i = \frac{1}{\rho_i}(\frac{\partial\sigma}{\partial x})_i + \sum_j m_j \Pi_{ij}.\frac{\partial w_{ij}}{\partial x} \qquad (22)$$

Another similar stabilizing term, which has been widely used in several applications of solid mechanics and hydrodynamics problems, has the following form: [7]

$$\pi = \begin{cases} \alpha.\rho.c.h.\dfrac{\partial v}{\partial x} + \beta.\rho.h^2.(\dfrac{\partial v}{\partial x})^2 & if\ \dfrac{\partial v}{\partial x} < 0 \\ 0 & otherwise \end{cases} \tag{23}$$

where α, β are constants. This stabilizing term is also added to the stress term in equilibrium equation,

$$(\frac{dv}{dt})_i = \frac{1}{\rho_i}(\frac{\partial(\sigma - \pi)}{\partial x})_i \tag{24}$$

4.2 Displacement based procedure using the Newmark's time integration

Combining the three sets of equations (18), (19), (20), allows derivation of the Navier equation, only based on displacement variables:

$$\rho.\ddot{u} = E\frac{\partial^2 u}{\partial x^2} \tag{25}$$

Equation (25) can be written in an incremental form:

$$\rho.\Delta\ddot{u}^n = E(\frac{\partial^2 \Delta u}{\partial x^2})^n \tag{26}$$

By using the Newmark's formulation, the incremental values of variables can be determined:

$$\begin{aligned} \Delta\ddot{u}^n &= (\frac{1}{\beta.(\Delta t)^2}\Delta u^n - \frac{1}{\beta.\Delta t}\dot{u}^n - \frac{1}{2\beta}\ddot{u}^n) \\ \Delta\dot{u}^n &= \frac{\gamma}{\beta.\Delta t}\Delta u^n - \frac{\gamma}{\beta}\dot{u}^n + \Delta t(1 - \frac{\gamma}{2\beta})\ddot{u}^n \end{aligned} \tag{27}$$

where β, γ are constants, taken as $\beta = 0.25, \gamma = 0.5$ for the average acceleration method , and $\beta = 1/6, \gamma = 0.5$ for the linear acceleration approach.

By using the updating procedure (27) at time step n and the CSPM methodology for evaluation of the second derivative of incremental displacement, the following numerical form of (26) is obtained,

$$\Delta u_i^n = \frac{\sum_{j=1}^{N}[(w_{ij}^a.\Delta x_j \frac{a_2}{a_1} - w_{ij}^s.\Delta x_j)\Delta u_j^n] - \frac{\rho a_3}{2E\beta.\Delta t}\dot{u}_i^n - \frac{\rho a_3}{4E\beta}\ddot{u}_i^n}{\sum_{j=1}^{N} w_{ij}^a.\Delta x_j \frac{a_2}{a_1} - \sum_{j=1}^{N} w_{ij}^s.\Delta x_j - \frac{\rho a_3}{2E\beta.\Delta t^2}} \tag{28}$$

with

$$a_1 = \sum_{j=1}^{N} w_{ij}^{a}.(x_j - x_i).\Delta x_j$$

$$a_2 = \sum_{j=1}^{N} w_{ij}^{s}.(x_j - x_i).\Delta x_j$$

$$a_3 = \sum_{j=1}^{N} w_{ij}^{s}.(x_j - x_i)^2.\Delta x_j$$

Any displacement boundary condition can be directly satisfied in (28). In case of existing force boundary condition, equation (28) has to be modified. By replacing $\Delta \ddot{u}^n$ from (27) into (26) and $(\frac{\partial^2 \Delta u}{\partial x^2})^n$ using (9) and considering that $E.\frac{\partial \Delta u}{\partial x}$ is equal to $\Delta\sigma$:

$$\Delta u_i^n = \frac{\sum_{j=1}^{N}[(-w_{ij}^{s}.\Delta x_j)\Delta u_j^n] - \frac{\rho a_3}{2E\beta.\Delta t}\dot{u}_i^n - \frac{\rho a_3}{4E\beta}\ddot{u}_i^n + \Delta\sigma_i^n a_2}{-\sum_{J=1}^{N} w_{ij}^{n}.\Delta x_j - \frac{\rho a_3}{2E\beta.\Delta t^2}} \tag{29}$$

where $\Delta\sigma_i^n$ is the force boundary condition.

It should be noted that both the Newmark's time integration method and the second derivative evaluation using CSPM, are based on Taylor series expansion with first three terms. Because of similarity between CSPM and FDM, truncation errors of (28) and (29) are nearly identical to the truncation error discussed in section 3. Therefore, by considering equation (17), it is calculated that the truncation error of the proposed procedure only constitutes the dissipation error and no dispersion will be induced in results.

Numerical tests have shown that the results of solving incremental displacement based equation using Newmark's time integration method and CSPM second derivative evaluation, can remove dispersion error, similar to other methods that eliminate dispersion using stabilizing terms. The dissipation error may also be removed by adding an appropriate term to the Navier equation.

5 NUMERICAL TESTS

In this section a standard problem is adopted to compare all following solution procedures:

a) stress based without modification using (18), (19), (20)
b) stress based using the first modification of section 3 and equations (21), (22)
c) stress based using the second modification of section 3 and equations (23), (24)
d) incremental displacement based, and the Newmark's time integration method using equations (28), (29)
e) incremental displacement based, and finite difference time integration method using equations (28), (29)

The problem is defined as one end fixed axial bar with 0.1 m length and 1.0 m^2 area of cross section, built of steel with elasticity modulus of 2.27e5 MPa, and density of 7800 kg/m^2 subjected to a 10^9 N point load applied to its free end.

Other solution parameters such as the time step, smoothing length and the number of particles are assumed similar in all cases.

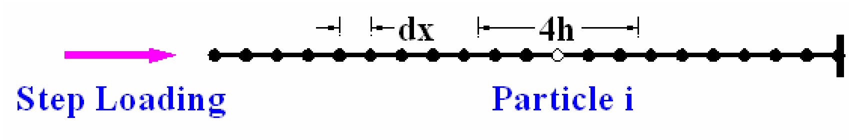

Figure 2: 1-D bar test

101 particles are assumed for the analysis. The time step is selected in a way that, the wave propagation can be observed in consecutive particles in successive time steps. Therefore, $\Delta t \leq \frac{\Delta x}{c_0}$, where Δx is the distance between two adjacent particles, and c_0 is the sound velocity in media; $c_0 = \sqrt{\frac{E}{\rho}}$. In the present simulations, Δt is taken as 5.0e-8 sec and the smoothing length has been chosen 0.001 m.

The time history of axial stress at the middle of the bar predicted by method (a) is depicted in Figure 3.

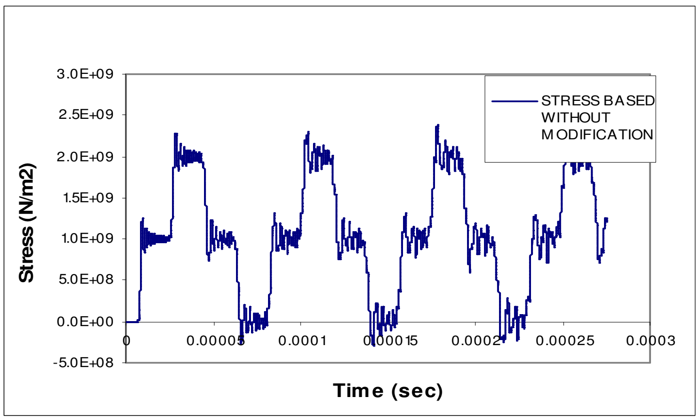

Figure 3: Stress time history of the middle span of the bar using stress based equation without modification

As explained in section 3, and can be clearly observed from Figure 3, these results contain truncation error in the form of dispersion.

The energy error can be defined as:

$$\begin{aligned} Energy_{error} &= Energy_{kinetic} + Energy_{potential} - Energy_{input} \\ Energy_{kinetic} &= \int \frac{1}{2} \rho v^2 d\Omega \\ Energy_{potential} &= \int \sigma . \varepsilon . d\Omega \\ Energy_{input} &= -u.F \end{aligned} \tag{30}$$

And is illustrated according to Figure 4:

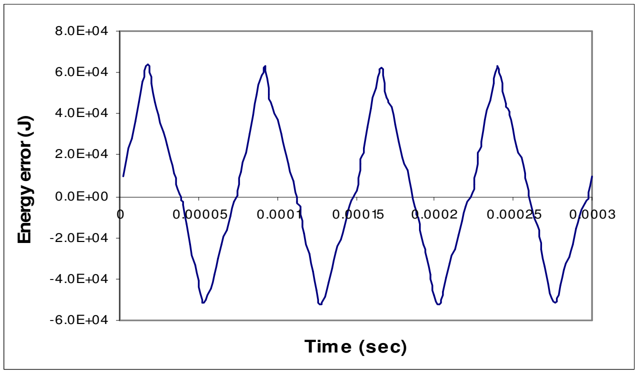

Figure 4: Energy error time history using stress based equation without modification

Although the overall energy loss is approximately zero, nevertheless, in order to remove instability of the method and its dispersion error, a more efficient and stable procedure such as (b), (c) or (d) should be adopted.

The results of the same problem using (b), (c), (d) methods are shown in Figure 5:

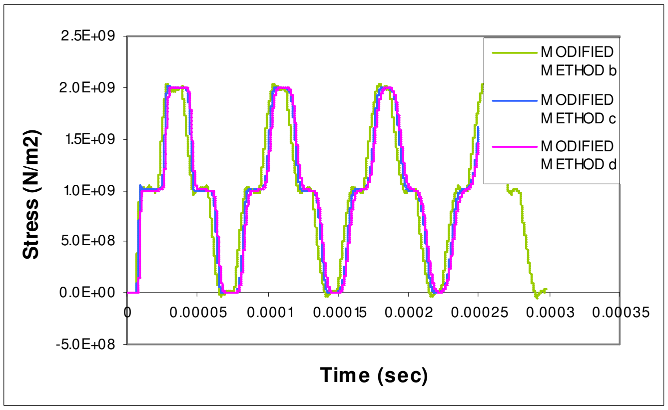

Figure 5: Stress time history of the middle span of bar using methods (b), (c), (d)

As expected, combining the displacement based equation and the Newmark's time integration, method (d), can remove dispersion error similar to methods using stabilizing terms.By computing the energy error using equation (30), the time history of error can be derived (Figure 6):

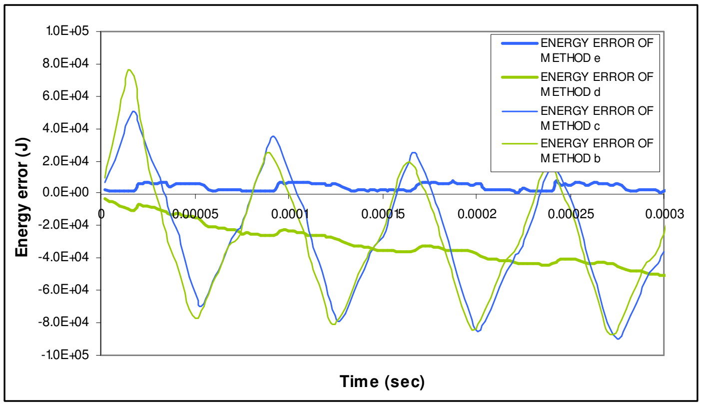

Figure 6: Energy error time history using methods (b), (c), (d), (e)

It is clear from Figure 6 that methods capable of removing dispersion, may contain some energy loss, and as a result some amplitude decay may appear in results (See Figure 5). The proposed method (d), however, has apparently less and smoother energy error.

Method (e) has the least energy loss (see Figure 6), but there remains some dispersion error as observed in its results (see Figure 7).

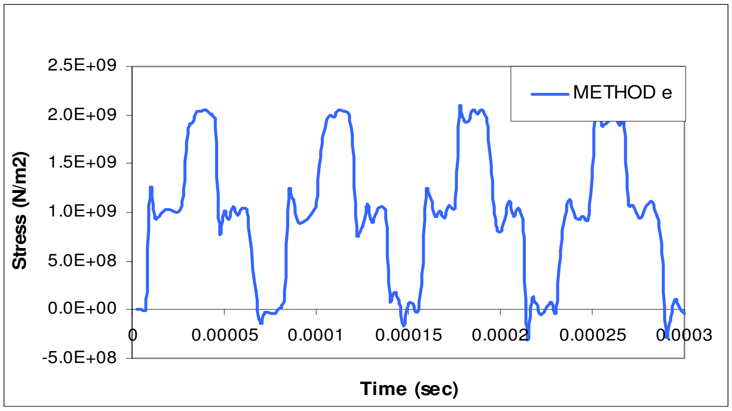

Figure 7: Stress time history of the middle span of bar using method (e)

6 CONCLUSION:

In this paper, the truncation error and its effect on solution of one dimensional problems were discussed and a quick review on methods preventing this drawback was provided In addition to existing methods, a new approach for dispersion reduction using incremental displacement based equation and the Newmark's time integration scheme has been proposed and compared with other methods.The proposed method has no necessity of adding any stabilizing term to the main equation, and its energy error is less than other stabilized methods based on artificial terms.

REFERENCES

[1] Chen JK, Beraun JE, Jih CJ, "An improvement for tensile instability in smoothed particle hydrodynamics", Comput. Mech. 23 (1999a) 279-287

[2] Chen JK, Beraun JE, Jih CJ, "Completeness of corrective smoothed particle method for linear elastodynamics", Comput. Mech. 24 (1999b) 273-285

[3]Chen JK, Beraun JE, Jih CJ, "A corrective smoothed particle method for transient elastoplastic dynamics", Comput. Mech. 27 (2001) 177-187

[4] M.B. Liu, G.R. Liu, Z. Zong, "Constructing smoothing functions in smoothed particle hydrodynamics with applications", Journal of Computational and applied Mathematics, 155 (2003) 263-284

[5] Gray J.P., Monaghan J.J., Swift R.P., "SPH elastic dynamics", Comput. Methods Appl. Mech. Engrg 190 (2001) 6641-6662

[6] W.K. Liu, S. Jun, Y.F. Zhang, "Reproducing kernel particle methods", J. Numer. Methods Eng. 20 (1995) 1081–1106.

[7] G. M. Zhang, R. C. Batra, "Modified smoothed particle hydrodynamics method and its application to transient problems", Computational Mechanics 34 (2004) 137–146

[8] Dmitri Kuzmin, "Analysis of numerical dissipation and dispersion", http://www.mathematik.uni-dortmund.de/~kuzmin/cfdintro/lecture10.pdf