ANALYTICAL SOLUTION FOR SHOCK WAVE PROPAGATION IN GRANULAR MATERIALS

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Abstract

Analytical solution of Shock wave propagation in pure gas is usually addressed in gas dynamics. However, such a solution for granular media is complex due to inclusion of parameters relation to particles configuration in the medium, called macroscopic parameters. In this paper, an analytical solution for gas flow in an isotropic homogenous granular material is presented. In this regard, balance equations are first written in waveform and then in non-conservation form. Afterwards, an equivalent gas is introduced by redefining thermodynamic properties. Finally, analytical solution is obtained using Riemann invariants along characteristics. The solution enables expressing shock propagation velocity, as well as density and pressure variations, in the porous medium in terms of gas properties and macroscopic parameters, which is of high value in design of granular shock isolators.

Keywords: shock wave, analytical solution, Riemann invariants, granular, gas flow, isolator

1. INTRODUCTION

Shock wave propagation modeling has always been a subject of high interest among researchers in many fields of engineering and science. This includes physical modeling using shock tube test device, as well as numerical modeling using various methods. In this regard, investigation of shock wave propagation in porous media is of crucial importance due to its application in military and industry.

Numerical shock simulations in porous media have gained quite popularity by the advent of high-resolution shock capturing techniques in computational fluid dynamics. Extensive reviews of the methods have been presented by Löhner (2008)[1], Hoffmann and Chiang (2000)[2] and Laney (1998)[3], among others.

Analytical methods of shock wave modeling in fluids are often limited to solution of the Riemann problem and generalized Riemann problem for Euler hyperbolic system of equations, in which method of characteristics is mainly utilized. Lax (1973) was one of the pioneers of the subject and provided the mathematical theory of shock waves. Toro (2009) and Guinot (2008)) have described the characteristics method of solving hyperbolic system of equations in detail and in an application-based style. They did not entrain rigorous closure conditions, such as equations of state for imperfect gases, in their textbooks. Such analytical solutions plus many others in viscid and inviscid fluid dynamics, including viscous boundary layers and Prandtl–Meyer flow, can be found in Emanuel (2000).

Analytical models for shock flow in rigid porous media have been presented by Levy et al. (1993, 1995a and 1995b), Krylov et al. (1996) and Sorek et al. (1996), in which significant evolution periods have been obtained by dimensional analyses of the governing balance equations of gas and solid phases after an abrupt change of thermodynamic properties of the fluid. The models supplied simplified governing equations for each period, but did not provide analytic solution similar to the one for the shock tube problem in fluids. Juanes and Patzek (2004) and Juanes (2005), for the first time, gave completely analytical solution for the Riemann Problem of three-phase flow in rigid porous media; however, they assumed incompressible fluids and considered only the conservation equations of mass into account using an extended multiphase form of the Darcy's law.

In this paper, analytical solution for one-dimensional weak shock wave flow through the pores of a rigid porous medium due to abrupt rupture of the diaphragm in the shock tube is proposed. It is assumed that the rigid granular material exists at both sides of the diaphragm. The solution is obtained using characteristic method by redefining thermodynamic properties of the gas inside the pores. Li et al. (1995) redefined the properties of the gas inside the pores and wrote the flow equations in primitive (physical) variable formulation of $[\rho_f \quad V_f \quad p_f]^T$ — ρ_f, V_f and p_f are density, velocity and pressure of the fluid, respectively—and declared that it can be applied in an analytical solution for shock wave propagation inside the porous medium; even though, they did not provide the solution based on their conservative formulation. However, it is clear that their formulation

is not mathematically true when shock waves are present and the solution is discontinuous. Toro (2009) has exemplified this and showed that this formulation is only mathematically true for smooth waves where rarefaction waves and contact discontinuities are present.

In the following sections, first the governing equations are described, and then the solution methodology is presented. Subsequently, veracity of the solution is verified by comparison with the numerical solution of the Rieman problem in rigid porous materials produced by Ben-Dor and Levy (1997).

2. GOVERNING EQUATIONS

In gas dynamics, one-dimensional Euler equations for a pure gas are expressed as

$$\frac{\partial \rho_g}{\partial t} + \frac{\partial (\rho_g V_g)}{\partial x} = 0$$
$$\frac{\partial (\rho_g V_g)}{\partial t} + \frac{\partial (\rho_g V_g^2 + p_g)}{\partial x} = 0 \tag{1}$$
$$\frac{\partial (E_g)}{\partial t} + \frac{\partial ((E_g + p_g)V_g)}{\partial x} = 0$$

The third equation in (1) is the energy equation, which becomes redundant in case of isothermal, polytropic and isentropic flows. In these cases, the equation of state for an ideal gas transforms to relations between pressure and density.

Weak shock waves are usually associated with low gas Mach numbers, M_g. From thermodynamics point of view, as a gas is forced through a tube, when ratio of the gas velocity, V_g, to the speed of sound in the gas, $a_g = \sqrt{\dfrac{\gamma p_g}{\rho_g}}$, is much less than unity ($M_g = V_g / a_g < 1$) in an adiabatic shock tube, the process maybe considered reversible; i.e., in view of the second law of thermodynamics, entropy of such a system is not increased and remains constant. Such a flow is an isentropic flow, in which the ideal (caloric) equation of state simplifies to

$$\frac{p_g}{\rho_g^\gamma} = C \tag{2}$$

Where γ is the specific gas constant and C is a constant depending on the type and temperature of the gas. Details on the thermodynamics can be found in any relevant source, e.g. Sonntag et al. (2003).

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In order to have an insight over the variations of the gas Mach number in shock tube test, we consider two shock tube tests: Sod (1978) test as a representative of weak shock wave and Woodward and Colella (1984) test as a representative of blast wave (Table 1). Gas Mach number for the two tests are depicted in Fig. 1 based on the classical analytical solution of the Riemann problem. It is clearly seen that the gas Mach number in the case of weak shock (Sod test) is well below unity, i.e., subsonic.

It is worth mentioning that the assumption of isentropic flow in granular media is not always the case, since heat transfer from gas to the solid particles plays great role on changing the flow type.

Table 1. Data for the two shock tube tests; 'L' and 'R' subscripts denote properties at the left and right of the diaphragm

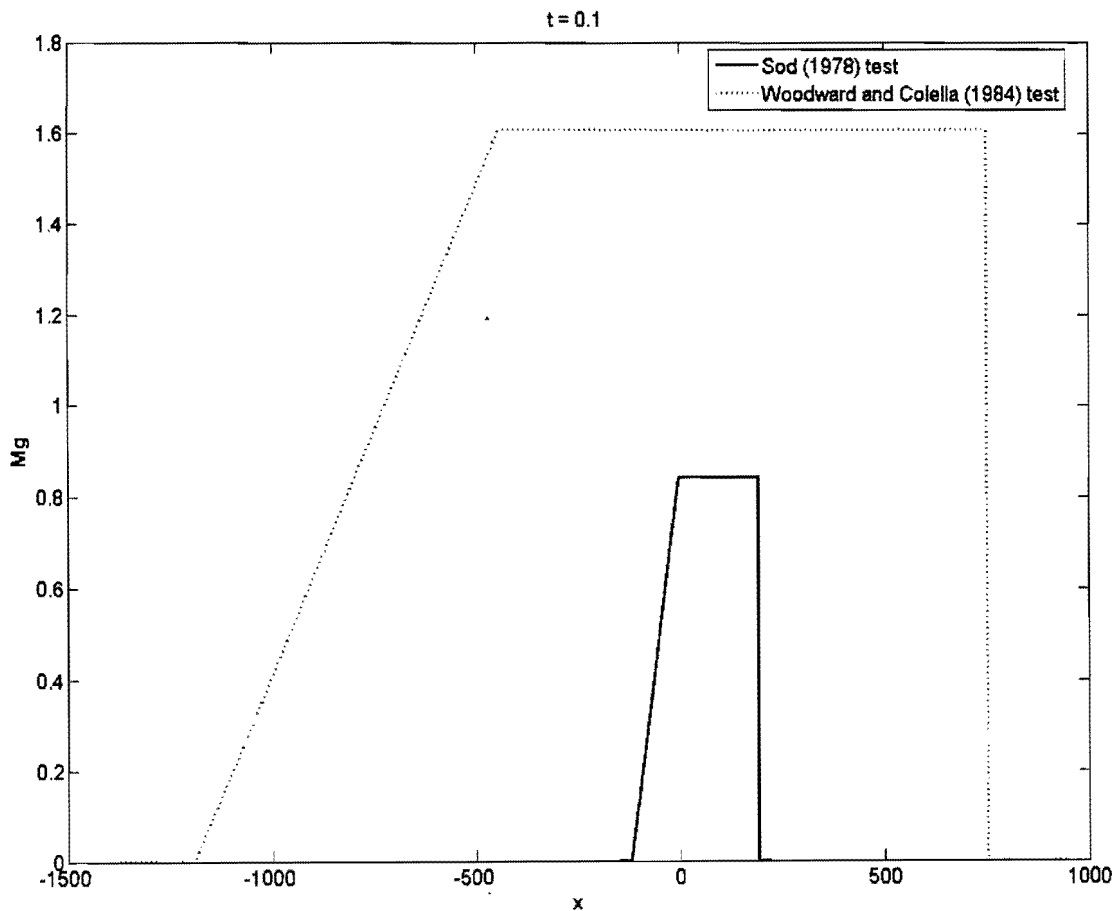| Test	ρ_{gL}	V_{gL}	p_{gL}	ρ_{gR}	V_{gR}	p_{gR}
Sod (1978)	1	0	1013250	0.1	0	101325
Woodward and Colella (1984)	1	0	101325000	1	0	101325

Fig. 1. Gas Mach number versus shock tube length for the two tests of Table 1, as representatives of weak and strong shock waves

Based on the above, the governing 1D Euler equations for an isentropic flow is written as

$$\frac{\partial \rho_g}{\partial t} + \frac{\partial \left(\rho_g V_g \right)}{\partial x} = 0$$
$$\frac{\partial \left(\rho_g V_g \right)}{\partial t} + \frac{\partial \left(\rho_g V_g^2 + p_g \right)}{\partial x} = 0$$

(3)

with the closure conditions specified in Eq.(2).

With the assumption of isentropic flow in rigid granular media in case of weak shocks, the Euler equations of the pure gas in Eq. (3) can be averaged over a Representative Elementary Volume (REV) to incorporate the effect of solid matrix on the gas flow inside the pores. Bear and Bachmat (1990) introduced an REV model that takes into account macroscopic (geometric) properties of the solid phase. The model is highly versatile and can be used in a variety of flow and transport phenomena. Present authors have further improved the model and derived completely analytical functions that excellently match empirical data for the macroscopic properties. The averaged flow equations are mentioned here without detailed manipulations and assumptions of the model, for which reference is made to Bear and Bachmat (1990):

$$\frac{\partial \rho_g}{\partial t} + \frac{\partial \left(\rho_g V_g \right)}{\partial x} = 0$$
$$\frac{\partial \left(\rho_g V_g \right)}{\partial t} + \frac{\partial \left(\rho_g V_g^2 + T_f^* p_g \right)}{\partial x} = 0$$

(4)

where T_f^* is the tortuosity scalar, which represents the tortuous flow path a gas particle traverses through the granular medium and its value is between zero and unity; the lower limit pertaining to a porosity of zero and the upper limit to the porosity of one. As the porosity increases, T_f^* also increases. Detailed discussion on this scalar and how it is related to the porosity of a homogenous granular medium is found in Ahmadi et al. (2010).

Observation of the Eq. (4) and its comparison with Eq. (3) reveals that for isentropic flow in rigid granular media, in which variations of porosity can be neglected, the solid phase interacts with the gas phase through modification of the pressure term in the momentum equation. Assuming $T_f^* = 1$ in Eq. (4) leads to the isentropic flow equations of (3) for the pure gas.

3. ANALYTICAL SOLUTION

To solve the system of equations of Eq. (4), we write it in conservation form:

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$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0 \qquad (5)$$

where

$$U = \begin{bmatrix} \rho_g & \rho_g V_g \end{bmatrix}^T \qquad (6)$$

is the vector of variables and

$$F = \begin{bmatrix} \rho_g V_g & \rho_g V_g^2 + T_g^* p_g \end{bmatrix} \qquad (7)$$

is the vector of fluxes.

To employ the characteristics method of solution on Eq. (5), it should be written in non-conservation (characteristic) form:

$$\frac{\partial U}{\partial t} + A(U) \frac{\partial U}{\partial x} = 0 \qquad (8)$$

where $A(U) = \dfrac{dF}{dU}$ is the Jacobian matrix.

To calculate the Jacobian matrix, we write

$$U = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T \qquad (9)$$

and using Eqs. (2) and (9),

$$F = \begin{bmatrix} u_2 & \dfrac{u_2^2}{u_1} + T_g^* C u_1^\gamma \end{bmatrix}^T \qquad (10)$$

Thus,

$$A = \begin{bmatrix} 0 & 1 \\ \dfrac{-u_2^2}{u_1^2} + \gamma T_g^* C u_1^{\gamma-1} & \dfrac{2u_2}{u_1} \end{bmatrix} \qquad (11)$$

After substituting for u_1 and u_2 from Eq. (6) in (11),

$$A = \begin{bmatrix} 0 & 1 \\ -V_g^2 + \dfrac{\gamma T_g^* p_g}{\rho_g} & 2V_g \end{bmatrix} \qquad (12)$$

The Jacobian matrix has the following two real distinct eigenvalues, and therefore is strictly hyperbolic:

$$\lambda = V_g \pm a_g \qquad (13)$$

in which a_f is the speed of sound inside the granular medium:

$$a_g = \sqrt{\dfrac{\gamma T_g^* p_g}{\rho_g}} \qquad (14)$$

Now, by defining an equivalent specific gas constant, $\gamma_{eq} = \gamma T_g^*$, and importing it in Eqs. (12) and (14), the system of equations pertaining to the gas flow inside the solid matrix transforms to the classical Riemann problem of gas dynamics in which a pure gas --with modified specific gas constant-- flows inside the shock tube.

According to the obtained eigenvalues in Eq. (13), both characteristic fields are genuinely nonlinear, i.e.

$$\nabla \lambda_i \cdot K^{(i)} \neq 0 \qquad i=1,2 \qquad (15)$$

where $K^{(i)}$ are the right eigenvectors corresponding to the eigenvalues.

Nonlinearity of the characteristic fields implies that there is no contact discontinuity present in the solution.

Rest of the procedure for finding the solution is straightforward and discussed thoroughly by many others, e.g. Toro (2009); thus, it will not be described here.

Based on the above equations and the classical analytical solution to the Riemann problem, a Matlab code was written, the results of which are used for verification in the next section.

4. Verification

As pointed out by Ben-Dor and Levy (1997), due to technical difficulty in rupturing the diaphragm that separates the high and low pressure chambers, in case of existence of granular medium at both sides of the diaphragm, the problem cannot be set up in a conventional shock tube. Therefore, it is unlikely to find experimental evidence to compare the results of the analytical solution.

However, in order to have a quantitative estimate of the proposed solution, reference is made to the one-dimensional numerical simulation by Ben-Dor and Levy (1997). Initial conditions in their simulations are specified in Table 2. Their simulation was not limited to the isentropic gas flow inside the pores, as contemplated in the present paper. Instead, they not only accounted for all of the six governing equations of the granular medium --three for the gas and three for the solid phase--, but also the source terms at the right-hand size of Eq.(5) that have been neglected by us by assuming Euler conditions. Porosity variations were also allowed in their study. An upwind TVD high-resolution shock-capturing scheme with second order of accuracy was utilized in their simulations.

Table 2. Initial conditions at the sides of the diaphragm in the numerical simulations of Ben-Dor and Levy (1997); 'n' and 'T' are the porosity and temperature of the granular medium

$P_{gL} = 1013250$	$P_{gR} = 101325$
$V_{gL} = V_{sL} = 0$	$V_{gR} = V_{sR} = 0$
$T_{gL} = T_{sL} = 300K$	$T_{gR} = T_{sR} = 300K$
$n_L = 0.73$	$n_R = 0.73$
$T_{gL}^{*} = 0.7$	$T_{gR}^{*} = 0.7$

The results of Ben-Dor and Levy (1997) study showed little porosity variations under the specified initial conditions. Gas Mach number diagram pertaining to their initial conditions in Fig. 1, represented by Sod test, is an indication of subsonic or weak shock wave flow. Consequently, the assumption of rigid porous medium and isentropic flow for their test conditions is reasonable to be applied in the present analytical model for verification purposes.
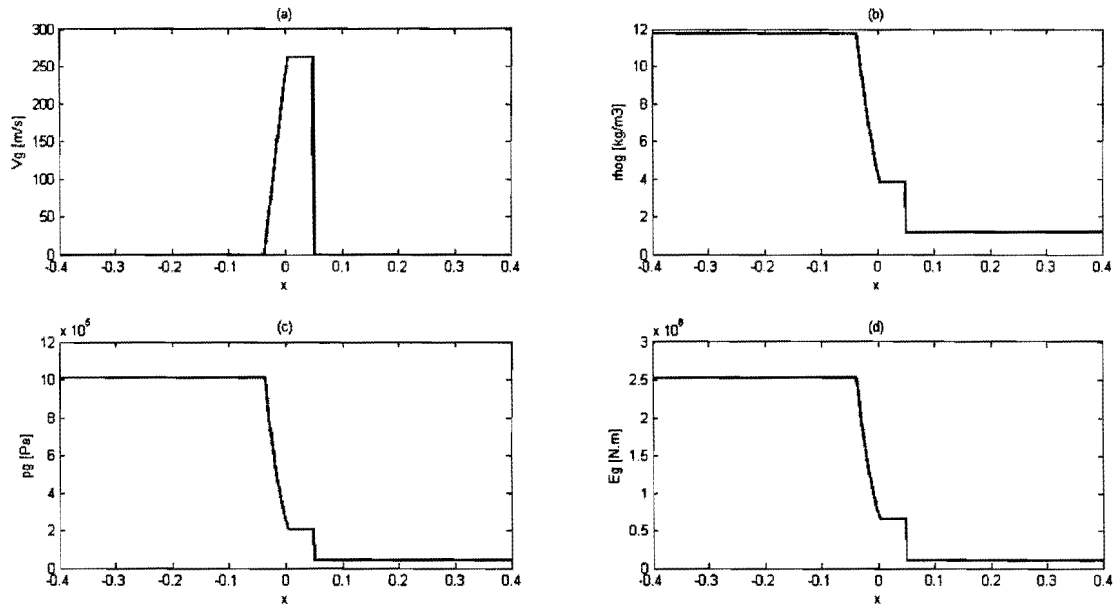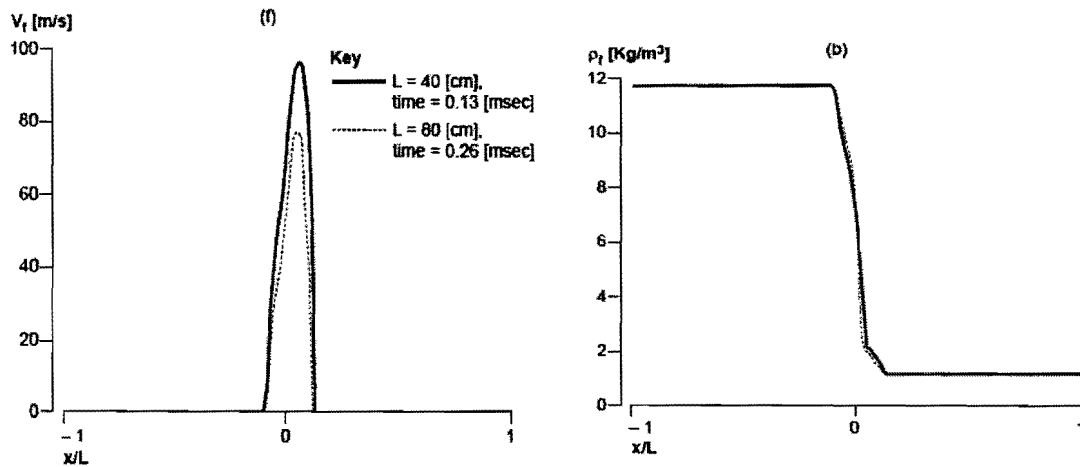
The results are shown in Fig. 2 and Fig. 3.

Fig. 2. Different diagrams obtained from the proposed analytical solution: gas (a) velocity (b) density (c) pressure (d) specific energy versus length of the shock tube at t=0.13 msec for a sample of L=40cm long; diaphragm is at x=0 before rupture
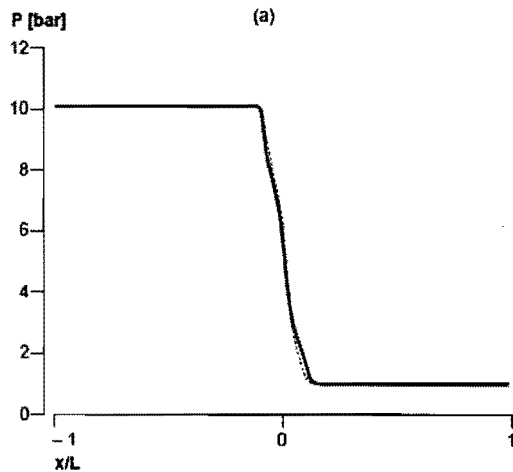

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Fig. 3. Different diagrams obtained from Ben-Dor and Levy (1997) simulation: gas (f) velocity (b) density (a) pressure versus length of the shock tube at t=0.13 msec for a sample of L=40cm long; diaphragm is at x=0 before rupture

Comparison of the figures reveals very good quantitative and qualitative coherence of the density and pressure diagrams from the analytical solution with the numerical simulation. This agreement is only qualitative for gas pressure velocity, while significant deviation is observed between the velocity values. At first sight, this may be ascribed to two factors. First is the fundamental difference in the systems of conservation equations in the two methods. While our simplified system of equations containing two conservation equations is strictly hyperbolic, the system of equations in Ben-Dor and Levy including six conservation equations is too complex to be solved analytically and is elliptic. Apart from this, the other significant factor is the inclusion of source terms in the numerical simulation. While Euler conditions contemplated in the present study prevents source terms such as gravitational effects, diffusion terms, etc to be incorporated, there is no such limit for numerical simulations.

5. Conclusion

In this paper, analytical solution for isentropic flow of gas inside rigid granular medium was proposed by utilizing the macroscopic equations of the gas averaged over an REV, writing the equations in characteristic form, defining an equivalent gas with modified

specific gas constant, and finally using the classical characteristic solution to the conventional Riemann problem in shock tube test.

Comparison with a rigorous numerical simulation of the similar problem showed very good qualitative and quantitative agreement of the results for gas pressure and density. The agreement for gas velocity was only qualitative, which can be attributed to difference in nature of the conservation equations, as well as contributions of source terms in the numerical simulation.

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