Extended finite element method in an orthotropic cracked medium

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Abstract

The problem of modeling crack in an orthotropic medium is considered. In this field, extended finite element method has been applied for modeling crack and analyzing the domain numerically. In this method, first finite element model without any discontinuities is made and then the two-dimensional asymptotic crack-tip displacement fields with a discontinuous function are added to the finite element approximation and enriched it using the framework of partition of unity. The advantage of this is the ability of taking into consideration the crack without any explicit meshing the crack surfaces. The mixed-mode stress intensity factors (SIFs) are evaluated to determine the fracture properties of domain and compare the results of proposed method with other numerical or (semi-) analytical methods. The SIFs are obtained by means of the form domain of interaction integral (M-integral).

Keywords: Orthotropic medium; Extended finite element method (X-FEM); Stress intensity factors

Introduction

The needs of modeling and analyzing of orthotropic materials have been revived of great interest recently, while there are enormous applications of such materials in various structural systems like those in aerospace and automobile industries and other advanced applied science. The main advantages of using these materials can be attributed to their high stiffness and low ratio of weight to strength in comparison to other materials.

Some analytical investigation have been reported on the fracture of such materials such as pioneering one by Muskelishvili [1] and other like Sih et al. [2], Tupholme [3], Viola et al. [4] and the later one like Lim et al. [5] and Nobile and Carloni [6].

There are many numerical method utilized in mechanical problems such as finite difference method, finite element method, discrete element analysis method, finite volume method and boundary element method. However, finite element method is more convenient and applicable because of its ability for modeling every boundary conditions, loadings, materials and geometries. To improve these abilities in order to model discontinuities, Blytschko et al. [7] combining FEM with partition of unity (proposed by Melenk and Babuška [8]) which known as eXtended Finite Element Method (XFEM). In the XFEM, finite element approximation enriched with appreciated functions extracted from fracture analysis around the crack-tip. The main advantage of the XFEM is modeling of discontinuities independently and the mesh prepared without any consideration about the discontinuities. In 2D isotropic media, by Moës et al [9] and Dolbow et al. [10] some studies have been reported.

In this study, the method proposed for one branch of orthotropic materials. The enriching functions are based on the work reported by Viola et al [4]. For the robustness of the proposed method stress intensity factors (SIFs) for a cracked plate are obtained by the method reported by Rao et al [11] and compared with other numerical or (semi-) analytical methods.

Mechanics of orthotropic materials

It is assumed that an infinite orthotropic plate consisting of a traction free line crack is subjected to uniform biaxial (T and kT) and shear (S) loads at infinity. Fig. 1 shows the crack geometry and loading and the Cartesian and polar co-ordinates utilized in this study.

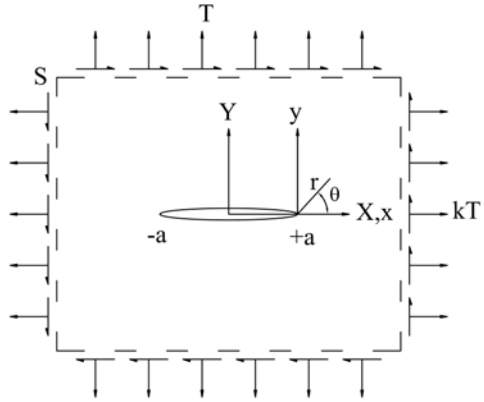

Figure 1. Crack geometry and loading and global and local co-ordinates conditions

In Viola et al [4] the elastodynamic case has been studied but in this paper the static case will be studied. As a result the velocity for the crack propagation assumed in Viola et al [4] equal to zero.

The displacement field in X(u) and Y(v) directions can be written [4]:

$$\begin{aligned} u = & -2\beta\left[\left(p_3 A_1 - p^4 B_1 + p_4 B_2\right)Y_1 + \left(p_3 B_1 + p_4 A_1\right)X_1 + p_3 B_2 X_2\right] \\ & + \frac{\beta T}{\mu_{12} D_1}\left\{\left(p_3 k_6 + p_4 k_5\right)\left[2\left(a + r\cos\theta\right) - \sqrt{2ar}\left(\sqrt{c_1(\theta)}\cos\theta_1/2 + \sqrt{c_2(\theta)}\cos\theta_2/2\right)\right]\right. \\ & \left. - \left(p_3 k_5 + p_4 k_5\right)\sqrt{2ar}\left(\sqrt{c_1(\theta)}\sin\theta_1/2 - \sqrt{c_2(\theta)}\sin\theta_2/2\right)\right\} \\ & + \frac{\beta S}{\mu_{12} D_1}\left\{\left(p_3 k_3 + p_4 k_4\right)\left[X_1 - X_2 + \sqrt{2ar}\left(\sqrt{c_2(\theta)}\cos\theta_2/2 - \sqrt{c_1(\theta)}\cos\theta_1/2\right)\right]\right. \\ & \left. - \left(p_3 k_4 + p_4 k_3\right)\left[2Y_1 - \sqrt{2ar}\left(\sqrt{c_1(\theta)}\sin\theta_1/2 + \sqrt{c_2(\theta)}\sin\theta_2/2\right)\right]\right\} \end{aligned} \tag{1}$$

$$
\begin{aligned}
v = &-\left[(\gamma_1 A_1 - \gamma_2 B_1 - \gamma_2 B_2)Y_1 + (\gamma_2 A_1 + \gamma_1 B_2)X_1 - \gamma_1 B_2 X_2\right] \\
&+ \frac{T}{2\mu_{12} D_1}\left\{(\gamma_1 k_6 + \gamma_2 k_5)\left[(X_1 - X_2) + \sqrt{2ar}\left(\sqrt{c_1(\theta)}\cos\theta_2/2 - \sqrt{2ar}\cos\theta_2/2\right)\right]\right. \\
&\left.+ (\gamma_1 k_5 - \gamma_2 k_6)\left[2Y_1 - \sqrt{2ar}\left(\sqrt{c_1(\theta)}\sin\theta_1/2 + \sqrt{c_2(\theta)}\sin\theta_2/2\right)\right]\right\} \\
&+ \frac{S}{2\mu_{12} D_1}\left\{(\gamma_1 k_3 - \gamma_2 k_4)\left[2(a + r\cos\theta) - \sqrt{2ar}\left(\sqrt{c_1(\theta)}\cos\theta_1/2 + \sqrt{c_2(\theta)}\cos\theta_2/2\right)\right]\right. \\
&\left.+ (\gamma_2 k_3 + \gamma_1 k_4)\sqrt{2ar}\left(\sqrt{c_1(\theta)}\sin\theta_1/2 + \sqrt{c_2(\theta)}\sin\theta_2/2\right)\right\}
\end{aligned} \tag{2}
$$

where p_j (j=1-4), k_j (j=1-6), A_1, B_1, B_2 and D_1 are ×constant loadings and material property coefficients and

$$X_1 = (a + r\cos\theta) - \gamma_1 l^2 r\sin\theta \tag{3-1}$$

$$X_2 = (a + r\cos\theta) + \gamma_1 l^2 r\sin\theta \tag{3-2}$$

$$Y_1 = \gamma_2 l^2 r\sin\theta \tag{3-3}$$

and

$$c_j(\theta) = \left(\cos^2\theta + l^2\sin^2\theta + (-1)^j l^2 \sin 2\theta\right)^{1/2}, \quad l^2 = \left(\gamma_1^2 + \gamma_2^2\right)^{-1} \tag{4}$$

$$\theta_j = arctg\left(\frac{\gamma_2 l^2 \sin\theta}{\cos\theta + (-1)^j \gamma_1 l^2 \sin\theta}\right), \quad j = 1,2. \tag{5}$$

$$\gamma_1 = \left[\frac{1}{2}\left(\frac{C_{22}}{C_{11}}\right)^2 + \frac{1}{4}\left(\frac{C_{22}}{C_{11}}\right) - \frac{1}{4}\left(\frac{C_{12}^2}{C_{11}C_{33}}\right) - \frac{1}{2}\left(\frac{C_{12}}{C_{11}}\right)\right]^{1/2} \tag{6-1}$$

$$\gamma_2 = \left[\frac{1}{2} + \left(\frac{C_{22}}{C_{11}}\right)^2 - \frac{1}{2}\left(\frac{C_{22}}{C_{33}}\right) + \frac{1}{2}\left(\frac{C_{12}^2}{C_{11}C_{33}}\right) + \left(\frac{C_{12}}{C_{11}}\right)\right]^{1/2} \tag{6-2}$$

where C_{ij} (i,j=1,2 and 3) are constitutive coefficients.
The displacements are limited to the case that γ_1 and γ_2 have real values as described in Viola et al [4].

Extended finite element Method

In XFEM the procedure of preparing the numerical analysis model is divided into two parts. In first part, the finite element model is made without any considerations about cracks, holes or other discontinuities and then by utilizing asymptotic near-tip functions and generalized Heaviside function the approximation used for displacement is enriched when the framework of partition of unity is applied.
If **x** is a point of a domain, in XFEM the approximation used for displacement can be written as

$$\mathbf{u}^h(\mathbf{x}) = \underbrace{\sum_{\substack{I \\ n_I \in \mathbf{N}}} \phi_I(\mathbf{x})\mathbf{u}_I}_{\text{classical part of FEM}} + \underbrace{\sum_{\substack{J \\ n_J \in \mathbf{N}^g}} \phi_J(\mathbf{x})\psi(\mathbf{x})\mathbf{a}_J}_{\text{enriched part of FEM}} \tag{7}$$

where $\mathbf{N}$ is a set of all nodes in the domain, $\mathbf{N}^g$ is a set of nodes that enriched with $\psi(\mathbf{x})$ function, $\mathbf{a}_J$ is a set of additional degree of freedom related to the discontinuities and ϕ_I is the finite element shape function.

For modeling an arbitrary crack Eq. (7) can be rewritten as below

$$\begin{aligned}\mathbf{u}^h(\mathbf{x}) = &\sum_{\substack{I \\ n_I \in \mathbf{N}}} \phi_I(\mathbf{x})\mathbf{u}_I + \sum_{\substack{J \\ n_J \in \mathbf{N}^g}} \mathbf{b}_J \phi_J(\mathbf{x}) H(\mathbf{x}) + \sum_{k \in \mathbf{K}^1} \phi_k(\mathbf{x}) \left(\sum_l \mathbf{c}_k^{l1} \mathrm{F}_l^1(\mathbf{x}) \right) \\ &+ \sum_{k \in \mathbf{K}^2} \phi_k(\mathbf{x}) \left(\sum_l \mathbf{c}_k^{l2} \mathrm{F}_l^2(\mathbf{x}) \right)\end{aligned} \tag{8}$$

where $H(\mathbf{x})$ is the generalized Heaviside function and defined by

$$H(\mathbf{x}) = \begin{cases} +1 & ;if\ (\mathbf{x} - \mathbf{x}^*).\mathbf{e_n} > 0 \\ -1 & ;otherwise \end{cases} \tag{9}$$

where $\mathbf{x}^*$ is the nearest point on the crack to $\mathbf{x}$ and $\mathbf{e_n}$ is the unit vector normal to the crack alignment.

In Eq. 8, $\mathrm{F}_l^1(\mathbf{x})$ and $\mathrm{F}_l^2(\mathbf{x})$ are near-tip enrichment functions and to derive them, it is noted that these functions must span the displacement fields in Eqs. (1) and (2); therefore one can write:

$$\{F_l(r,\theta)\}_{l=1}^4 = \left\{ \sqrt{r}\cos\frac{\theta_1}{2}\sqrt{g_1(\theta)}, \sqrt{r}\cos\frac{\theta_2}{2}\sqrt{g_2(\theta)}, \sqrt{r}\sin\frac{\theta_1}{2}\sqrt{g_1(\theta)}, \sqrt{r}\sin\frac{\theta_2}{2}\sqrt{g_2(\theta)} \right\} \tag{10}$$

Example

In this example proposed method is applied to a slanted crack of length 2a located in a finite two-dimensional orthotropic plate under constant applied tension (Fig. 2) where $2a = 2\sqrt{2}$.

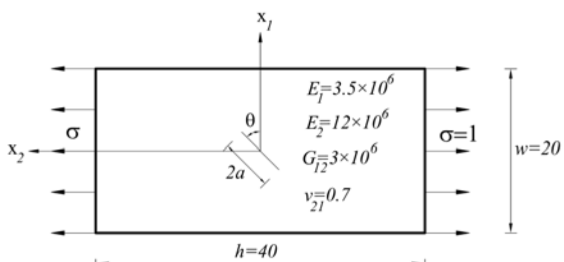

Figure 2. geometry for plate with a slanted crack under remote tension

Stress intensity factors are evaluated and compared with results reported by Sih et al [2], Alturi et al [11], Wang et al [12] and Kim and Paulino [13] and provided in table 1. As seen in table 1 the result are different 2.6% for K_I and 3.6% for K_I related to Sih et al [2].

Method		K_I	K_{II}
Sih et al [2]		1.0539	1.0539
Atluri et al. [11]		1.0195	1.0795
Wang et al. [12]		1.023	1.049
Kim and Paulino [13]	MCC	1.067	1.044
	DCT	1.077	1.035
Proposed method		1.081	1.092

Table 1. SIFs in an orthotropic plate with a slanted crack under uniform remote tension loading

Conclusion

In this paper an extended finite element method is proposed for analyzing cracked orthotropic materials. In this study, analytical displacement field around a crack-tip in orthotropic media is used to extract near-tip enrichment functions. In the present approach, a set of partition of unity based enriching functions are added to finite element approximation so the crack geometry can be taken into account without any special meshing. The robustness of suggested method was tested with evaluating stress intensity factors and comparing them by other available methods and they are in good agreement with other numerical and semi-analytical method.

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