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Improving time integration method of CSPM meshless method in elastodynamic problems

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Abstract: *Conventional time integration procedure for particle methods is the finite difference method. This simple procedure may generate numerical instability. To avoid this drawback and to increase accuracy of the scheme, small time steps must be used causing an increase in analysis time. In this paper, a numerical form of Navier equations of two dimensional elastodynmic is presented that allows using more reliable and stable time integration procedures such as Newmark's method. The corrective smoothed particle method (CSPM) that was firstly presented by J.K. Chen in 1999 is adopted as the particle method. A wave propagation in an elastic bar example is analyzed with conventional time integration and the proposed method.*

1 Introduction

Smoothed particle hydrodynamics (SPH) method was developed firstly by Gingold, Monaghan and independently by Lucy in astrophysical problems. But it encountered some physical and mathematical problems. One of the most important problems in standard SPH is its lack of consistency in boundaries. Another drawback is its failure in satisfying boundary conditions. In addition to early remedies of these drawbacks, Chen in 1999 [1] proposed a new method by using taylor expansion series which can satisfy consistency conditions needs for second order problems, and boundary conditions can be directly applied. One of important problems that must be taken in to attention, is the time integration method. The conventional time integration method for particles meshless methods is the finite difference which may induce numerical instability in solutions with longer time steps. In usual particles form of equation, only first order time-derivatives of variables are involved and only the F.D.M. can be used. For applying different methods of time integration, different form of equations are required. In this paper, the Newmark's time integration is used for Navier equations for improving results of CSPM method.

2 CSPM stress based form of 1-D elastodynamic equations:

In this method, the following three sets of equations are written in strong form:

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Equilibrium equations in step i:

$$(\rho\dot{v})^i = (\frac{\partial \sigma}{\partial x})^i \tag{1}$$

Compatibility equations in step i:

$$(\dot{\varepsilon})^i = (\frac{\partial \dot{u}}{\partial x})^i \tag{2}$$

Constitutive relations for plane stress in step i:

$$(\dot{\sigma})^i = E.(\dot{\varepsilon})^i \tag{3}$$

Derivatives of displacement and stress variables will be calculated by CSPM method [2].
Equations (1), (2) and (3) are considered to be true in each time step separately. The procedure starts by applying the initial conditions and can be arranged as:

1. Calculate acceleration components in each time step, by applying equation (1) and initial conditions.
2. Obtain velocity components by F.D.M..
3. Calculate the strains rate by using compatibility equations.
4. By using F.D.M., the strain tensor can be calculated.
5. The stress tensor required at the start of next step is obtained from constitutive equation.
6. Repeat the procedure.

It is obvious from the above algorithm that the finite difference time integration is repeated twice in each time step. This repetition will generate more accumulative error in numerical procedure which appears as numerical instability for solutions with longer time steps. To illustrate this problem consider a step load is suddenly applied to one end of a 1m long bar that is fixed at another end with intensity of 10.E9 Mp. The cross sectional area of the bar is 1.0m x 1.0m square. This problem is solved by three different time steps, it is clearly obvious from displacement curve of end point of bar in figure (1) that by selecting a smaller time step numerical instability will appear in results.

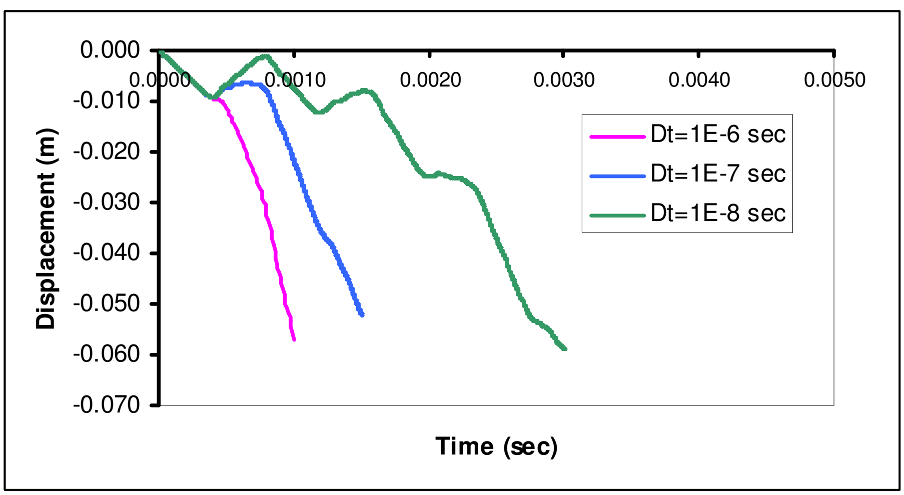

Figure 1: *Bar under step axial loading with three different time steps by F.D.M*
Above example has been solved by 21 particles in equal distance of 0 .05 m.

3 CSPM form of 1D Navier formulation and Newmark's time integration method:

Combining three sets of equations enables us to derive an equation, called Navier equation only based on displacement variables:

$$\rho\ddot{u} = E\frac{\partial^2 u}{\partial x^2} \tag{4}$$

Calculation of the second derivatives can then be performed by CSPM method by using taylor expansion series of "u" function:

$$u(x_1,x_2,x_3) = (u(x_1,x_2,x_3))_I + (\frac{\partial u}{\partial x_\alpha})_I.(x_\alpha - (x_\alpha)_I) + 1/2(\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta})_I.(x_\alpha - (x_\alpha)_I)(x_\beta - (x_\beta)_I) + R_2$$
(5)

Multiplying (5) by the weight function and its two first and three second derivatives and after integration over domain of the problem we obtain six equations are obtained with six unknown variables including function values and values of the first and second derivatives at point I.

$$\begin{aligned}
&\int (u-u_I).wd\Omega = \int (\frac{\partial u}{\partial x})_I.(x_\alpha - (x_\alpha)_I)wd\Omega + 1/2\int (\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta})_I.(x_\alpha - (x_\alpha)_I)(x_\beta - (x_\beta)_I)wd\Omega \\
&\int (u-u_I).\frac{\partial w}{\partial x_\alpha}d\Omega = \int (\frac{\partial u}{\partial x})_I.(x_\alpha - (x_\alpha)_I)\frac{\partial w}{\partial x_\alpha}d\Omega + 1/2\int (\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta})_I.(x_\alpha - (x_\alpha)_I)(x_\beta - (x_\beta)_I)\frac{\partial w}{\partial x_\alpha}d\Omega \\
&\int (u-u_I).\frac{\partial^2 w}{\partial x_\alpha \partial x_\beta}d\Omega = \int (\frac{\partial u}{\partial x})_I.(x_\alpha - (x_\alpha)_I)\frac{\partial^2 w}{\partial x_\alpha \partial x_\beta}d\Omega + 1/2\int (\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta})_I.(x_\alpha - (x_\alpha)_I)(x_\beta - (x_\beta)_I)\frac{\partial^2 w}{\partial x_\alpha \partial x_\beta}d\Omega
\end{aligned} \tag{6}$$

Equation (4) can be written in incremental form:

$$\rho.\Delta\ddot{u}^i = E\frac{\partial^2 \Delta u^i}{\partial x^2} \tag{7}$$

Now the Newmark’s method can be used for updating variables.

3-1 Updating values:

By using Newmark's formulation, the incremental values of variables are determined:

$$\begin{aligned}
\Delta\ddot{u}^i &= \frac{1}{\beta.(\Delta t)^2}\Delta u^i - \frac{1}{\beta.\Delta t}\dot{u}^i - \frac{1}{2\beta}\ddot{u}^i \\
\Delta\dot{u}^i &= \frac{\gamma}{\beta.\Delta t}\Delta u^i - \frac{\gamma}{\beta}\dot{u}^i + \Delta t(1 - \frac{\gamma}{2\beta})\ddot{u}^i
\end{aligned} \tag{8}$$

Substituting (8) in (7) will give the final incremental equation with only one unknown variable; Δu^i in step I.

In comparison with F.D.M., the Newmark's method is more stable but in longer time steps, accuracy will decrease. Lack of accuracy appears in form of amplitude decay and period elongation.
Another advantage of using Navier equation is higher accuracy with lower resolution of particles.
One important different in comparison of stress based and displacement based procedure is that the former formulation that is based on first derivative of variables causes numerical dispersion in solution and the latter that is based on second derivatives of variables causes numerical dissipation.

4 CSPM form of 2D Navier formulation and Newmark's time integration method:

Combining three sets of equations in two dimensional problems enables us to derive Navier equations that is only based on displacement variables:

$$\begin{aligned}
\rho\ddot{u}_x &= \frac{E}{(1-\nu^2)}\frac{\partial^2 u_x}{\partial x^2} + \frac{E}{2(1+\nu)}\frac{\partial^2 u_x}{\partial y^2} + \frac{E}{2(1-\nu)}\frac{\partial^2 u_y}{\partial x \partial y} \\
\rho\ddot{u}_y &= \frac{E}{(1-\nu^2)}\frac{\partial^2 u_y}{\partial y^2} + \frac{E}{2(1+\nu)}\frac{\partial^2 u_y}{\partial x^2} + \frac{E}{2(1-\nu)}\frac{\partial^2 u_x}{\partial x \partial y}
\end{aligned} \tag{9}$$

Calculation of the second derivatives can then be performed by (6) in x and y directions separately.
By using Newmark's time integration relations (8), equation (9) can be written in incremental forms:

$$\begin{aligned}
\rho.\left(\frac{1}{\beta.(\Delta t)^2}\Delta u_x^i - \frac{1}{\beta.\Delta t}\dot{u}_x^i - \frac{1}{2\beta}\ddot{u}_x^i\right) &= \frac{E}{(1-\nu^2)}\frac{\partial^2 \Delta u_x^i}{\partial x^2} + \frac{E}{2(1+\nu)}\frac{\partial^2 \Delta u_x^i}{\partial y^2} + \frac{E}{2(1-\nu)}\frac{\partial^2 \Delta u_y^i}{\partial x \partial y} \\
\rho.\left(\frac{1}{\beta.(\Delta t)^2}\Delta u_y^i - \frac{1}{\beta.\Delta t}\dot{u}_y^i - \frac{1}{2\beta}\ddot{u}_y^i\right) &= \frac{E}{(1-\nu^2)}\frac{\partial^2 \Delta u_y^i}{\partial y^2} + \frac{E}{2(1+\nu)}\frac{\partial^2 \Delta u_y^i}{\partial x^2} + \frac{E}{2(1-\nu)}\frac{\partial^2 \Delta u_x^i}{\partial x \partial y}
\end{aligned}$$
(12)

Final incremental equations only has two unknown variables in step I; Δu_x^i, Δu_y^i .

5 Numerical tests

The same example in section 2 has been analyzed by proposed method with different time steps. It is obviously seen that proposed method is stable in solutions with longer time steps and also it is clear by increasing time step accuracy will decrease. (Figure 2)

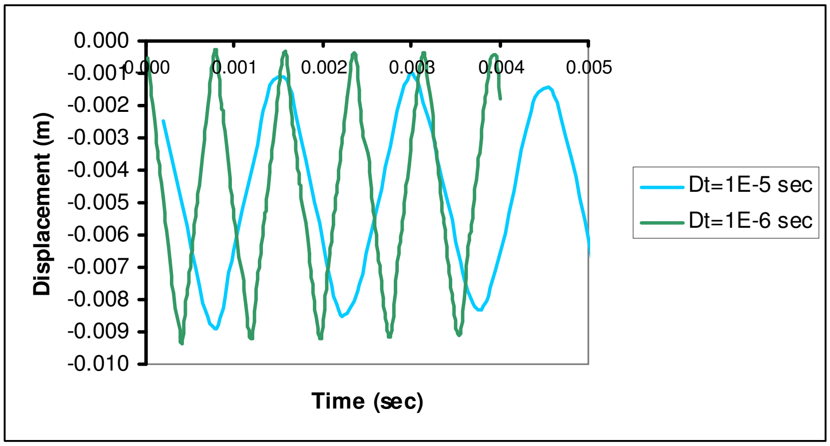

Figure 2: *Bar under step axial loading with two different time steps by Newmark's time integration method*

In this example also 21 particles is used to modeling the one meter long bar. Numerical dissipation in displacement based solution can be shown in axial stress time history curve of middle span. (Figure3)

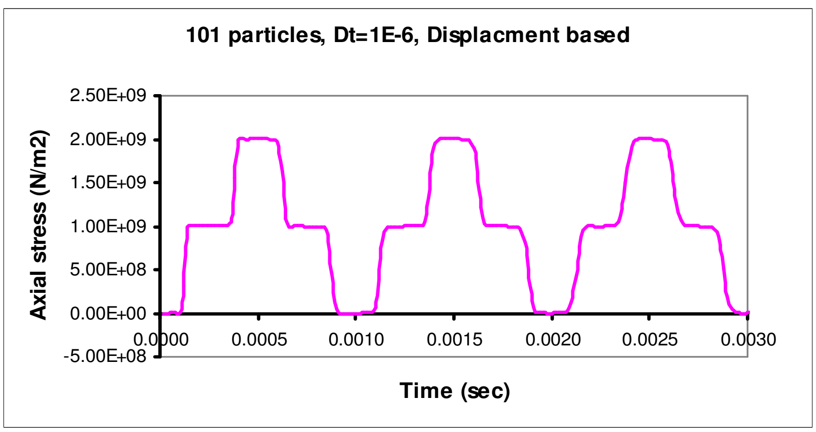

Figure 3: *Stress time history for displacement based solution*

Numerical dispersion in stress based solution can be shown in axial stress time history curve of middle span. (Figure4)

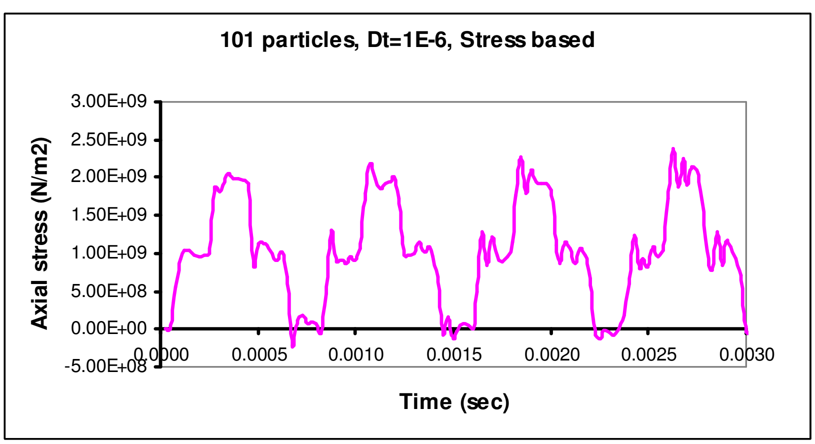

Figure 4: *Stress time history for Stress based solution*

6 Conclusion:

Combining equilibrium and compatibility equations and constitutive relations results in equations that are based only on displacement variables. More reliable and efficient time integration methods such as Newmark's method can apply for updating variables.

References

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