A local PUFEM modeling of stress singularity in sliding contact with minimal enrichment for direct evaluation of generalized stress intensity factors

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ABSTRACT

The order of stress singularity around sharp corners is studied by solving the characteristic equation numerically. The corresponding displacement and stress fields around the sharp corners, which accurately satisfy the compatibility of deformation and stress states on the two sides of the slave corner, are derived for various contact configurations. The dominant mode of infinite asymptotic stress field for contact problems is then implemented with minimum enrichments (2 functions for each enriched node), for the first time, within the framework of partition of unity finite element. An increased rate of convergence is achieved and the generalized stress intensity factor can be obtained directly from the additional unknowns. Numerical examples demonstrate the superior accuracy of the present approach to capture the sliding contact stress singularities near sharp corners.

1. Introduction

The study of stresses around the corner of composite wedges in contact is important in many industrial and research practices [1]. Among them, various indentation tests are commonly used to estimate the mechanical properties of thin films, surface coated specimen and bulk materials. These mechanical properties comprise elastic modulus, Poisson’s ratio, yield strength, stress hardening, power-low and/or other plastic, viscous and dynamic constitutive parameters. On the other hand, indentation process is present as a component of many manufacturing procedures which leads to conversion of a material from a primary form into more valuable productions through the use of mechanical processes such as rolling, forging, extrusion, coining, drawing and molding.

In the structural engineering field welded joints which comprise in-plane or out of plane indentation of sharp edges on an adhesive interface with a plate, may experience stress singularities which are a major source for potential failures. Despite the fact that indentation of sharp corners are known to cause problems and therefore are usually avoided in engineering design, there are some situations in industry where contact or impact of non-conforming interfaces are inevitable and thus an accurate estimation of the intensity of the local stress field is valuable for appropriate design, fabrication and maintenance of industrial instruments. Complete sliding contact of a punch in screw press cycles, ring rolling and cutting in metal forming processes are among such practical applications. Pertinent singularities are associated with different configuration of contact of dissimilar materials with different interfacial boundary conditions. In addition, fretting fatigue and crack nucleation and growth in the vicinity of the contact zone due to alternating high stress gradient may also be of interest.

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Nomenclature

- \( a \): vector of additional degrees of freedom
- \( a^e \): additional degrees of freedom associated to the enrichment function \( F_e(x) \) and compact support of \( N_i(x) \)
- \( A, B, C, D, A', B', C', D' \): parameters of the eigenfunction expansion
- \( A_0, B_0, C_0, D_0 \): parameters of the eigenfunction for the jth wedge
- \( A_i^e, B_i^e, C_i^e, D_i^e \): auxiliary asymptotic field parameters
- \( B(X) \): strain–displacement matrix for the enriched degrees of freedom
- \( d \): dimension of the block
- \( DTOL \): user defined tolerance for error control parameter
- \( F, F(\lambda, \theta) \): radial term of the Airy stress function
- \( F^\text{aux} \): external load vector component at the ith increment
- \( \bar{F}(x), \bar{F}(X) \): enrichment function in local x and y coordinates of the singular point
- \( \bar{F}_q(x) \): qth enrichment function
- \( g_1(\theta), g_2(\theta) \): first and second terms of the angular modes of the eigenfunction expansion
- \( g_i(\theta) \): angular term of the eigenfunction expansion for the jth component of displacement vector
- \( G_i(\lambda, \lambda') \): angular term of the Airy stress function for the jth wedge
- \( K \): coefficients of the ith term of eigenfunction expansion
- \( K_i \): GSIF for the first and second terms of the eigenfunction expansion
- \( K_{GSIF}, K_{II} \): first and second terms of the angular modes of the eigenfunction expansion for double root solutions
- \( K_{GSIF} \): generalized stress intensity factor
- \( N_i(x) \): finite element interpolation function
- \( r, \theta \): radius and angle
- \( r_e \): radius of enrichment domain
- \( r_0 \): radius of the contour integral path
- \( R \): error control parameter
- \( t_i, t_i' \): primary and auxiliary traction vector components
- \( U \): asymptotic displacement field
- \( u_i(x) \): finite element displacement approximation
- \( u_i \): displacement vector components
- \( u_i^a \): auxiliary displacement vector
- \( \Delta_x, \Delta_y, \Delta_i \): eigen-equation and its real and imaginary parts
- \( \Delta T_i \): load step size for the ith increment
- \( \Delta U_1, \Delta U_2 \): first and second orders of displacement increment
- \( \mu_f \): coefficient of friction
- \( \mu_y \): shear modulus for the jth wedge
- \( \kappa \): Kolosov’s constant
- \( \lambda, \lambda' \): singularity power, eigenvalue associated to the auxiliary field
- \( \sigma_{yy} \): primary stress tensor
- \( \xi, \eta \): real and imaginary parts of the singularity power
- \( \varepsilon(X) \): strain contribution of enrichment
- \( \phi_i, \phi_{s}, \phi_{II} \): Airy stress function, first and second terms of eigenfunction expansion
- \( \phi_{log} \): logarithmic Airy stress function for \( \lambda = \lambda_i \)
- \( \Gamma = \Gamma_1 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_2 \): contour integral path
- \( \Omega \): auxiliary stress tensor
- \( \Phi_j \): Airy stress function for the jth wedge
- \( \Sigma \): asymptotic stress field

Owing to the fact that the form of stress field near the singular points is independent of the far-field boundary conditions, the method of complex variables and semi-analytical methods are capable of determining the complete nature of the local high gradient or singular solution. Among these approaches are the complex function representation and the Mellin transform technique by Sternberg and Koiter [2], originally proposed for local mode expansion on the interface of composite bodies in linear elasticity. A semi-analytical approach that has been frequently used in the literature is the eigenfunction expansion method. The approach gives the opportunity to capture the self-similar higher order displacement fields near the point of transition of slip–stick boundary conditions in frictional contact interface or singular fields near the edge indentation or penetration of a sliding cone (i.e. a pile tip penetrating into the soil). The method indicates noticeable advantages in modeling a number of important engineering problems: interfacial contact along crack faces, missiles penetrating dissimilar deformable media and frictional conforming contact problems [3]. Brahtz appears to be the first to report the existence of...
contact corner singularities in 1933 [4], Williams carried out the first systematic study of singularities near corners with free–free, fixed–fixed and free–fixed boundary conditions [5]. The work of Williams was further extended by Karal and Karp [6], Kalandiia [7], England [8] and Vasilopoulos [9–11], among the others. Sternberg and Koiter studied the multiple roots in the Williams eigenfunction expansion [2]. Later, Dempsey and Sinclair obtained the logarithmic non-separable solution for singular wedge problems from the derivative of the Williams solution with respect to the singularity power for the first time [12]. Joseph and Zhang [13] discussed the multiple root solutions and singular stress states that were not variable–separable, following the method of Frobenius. Dini et al. dealt with inhomogeneous boundary conditions such as “patching in” a solution for the effect of the rounding of the sharp corner (e.g. boundary condition $\partial u_i/\partial r = cr$, at free edges of a wedge) [14].

Several studies have concentrated on different material behavior (e.g. anisotropic, strain hardening or a variety of multi-material elastic composites) and the possibility of a logarithmic singularity. The asymptotic singular fields for many adhesive joint geometries are not available due to the complexity of the analytical formulations. The length of resulting expressions for the determinant form eigenfunction equation, which gives the singularity or high intensity power, increases very fast when the number of involved materials is increased or there is a diversity in the boundary conditions on different interfaces. For instance, in bi-materials it is about one page expression; for three materials the expression of the determinant reaches to about 15 pages and for four materials it may even exhaust the capability of many computers [15]; therefore, the compact form of the characteristic equations, if it can be found, is very useful for computation and analysis. Particularly in recent years, the use of these eigen-functions to better reproduce the local character of the solution around singular points has become important in computational mechanics. Once the nature of the stress singularity is determined analytically, the corresponding asymptotic displacement fields may be used as shape functions to enrich the solution field in the vicinity of the contact corner by PUFEM or XFEM frameworks [16].

Contact mechanics literature incorporates numerous algorithms that deal with interaction of different fluids and/or structures. A three-dimensional extended EFG method coupled with FEM was utilized for modeling cohesive crack growth in reinforced concrete structures [17]. Bordas et al. [18] introduced a three-dimensional extrinsically enriched EFG for treatment of evolving multiple cracks in nonlinear solids including large deformations without near-tip enrichment. They proposed the construction of a Lagrange multiplier field in lieu of near-tip enrichment or influence domain adjustment to close the cracks along their fronts. A Meshfree thin shell model based on the Kirchhoff–Love theory was also adopted for nonlinear analysis of arbitrary evolving cracks [19]. Another method for treating crack growth in 3D particle methods by introducing the failure direction at individual particles based on a mixed Lagrangian–Eulerian kernel formulation was proposed by Rabczuk and Belytschko [20,21]. A local partition of unity enriched EFG method to simulate crack nucleation, propagation and joining cracks in nonlinear three dimensional solids was presented in [22], while Rabczuk et al. [23] proposed “Immersed Particle method” as a master–slave Lagrangian meshfree contact algorithm for dealing fluid–structure interactions. The recent work of Menk and Bordas [24] also determined numerically the order of singularity for some asymptotic problems of strain singularity such as junction points including a polycrystalline structure based on the approach developed by Li et al. [25]. Though well applicable for asymptotic problems especially in anisotropic materials, the proposed algorithm was merely an approximation of the singular field, with dependence on the angular discretization density around the singular point.

The state of stress singularity around bonded and sliding punch corners has been studied in both qualitative and quantitative analysis categories. The outline of the stress field near the corner vertex has been examined in a qualitative way via isochromatic patterns in some photo-elasticity experiments, which approved the existence of stress singularity for both lubricated and dry interfaces [26]. Very good agreements have been reported between the experimental and analytical solu-

Fig. 1. A typical contour of asymptotic shear stress $\sigma_{xy}$ obtained by the present study in comparison with an isochromatic view (upper left) in the vicinity of a sharp corner for a frictionless lubricated interface angle ($\phi \approx 77.5^\circ$) of identical materials at which singularity starts to appear [26].
tions on the shape of eigenfunction and the results of asymptotic analysis [27,28], as depicted typically in Fig. 1 in the vicinity of a sharp corner for a sliding steel punch on a steel half-plane.

Analytical investigation on the distribution of elastic fields near the end of the contact zone for the bonded punch corner is due to Williams [5], while the characteristic equation for sticking punch was solved by Gdoutos and Theocaris [29] and Comninou [30]. The obtained characteristic roots, i.e. the singularity power, form the eigen-functions which satisfy the boundary conditions and reproduce the singular stress field [16,26].

The numerical solution of eigenvalue problem for general interface boundary conditions in linear elasticity was originally proposed by Lee and Barber [27]. However, since the method assigns only one angle to each contact interface, then for example the problem of antisymmetric sliding around a corner cannot be studied. Asymptotic fields near the singular points in two dimensional orthotropic elastic media and the explicit form of the eigenequation of bonded orthotropic edge with an arbitrary wedge angle to an isotropic half-plane have been addressed in Wu and Liu [31]. Solving the characteristic equation numerically, gives the opportunity to capture singular stress fields in contact of orthotropic composite targets with the present approach.

Developing new enrichment functions to capture a local behavior of the solution has been frequently used as an efficient way of simulation for fracture problems [32–38]. In this paper, however, the singularity due to various frictional contact problems is considered and the corresponding enrichment functions are derived.

Enrichment functions have already been proposed in the framework of the asymptotic eigenfunction expansion in the literature for slipping and sticking punch indentation and bonded sides corner penetration in dissimilar materials [39]. In the next section, node-to-segment (NTS) slide line formulation will be enriched to reproduce the superior modes of stress singularity. The strength of singularity and the relative asymptotic parameters are determined in terms of the specified master and slave material elastic modules, corner angles and coefficients of friction. In contrast to Giner et al. [16], who studied the same case of sliding punch, the number of additional degrees of freedom is reduced as a result of combining the basic asymptotic modes in order to satisfy the characteristic equation by determining the field parameters in the pre-processing phase. The enrichment functions derived from the biharmonic equation of elasticity are the solution to several asymptotic problems with different boundary conditions, but with the same eigen-value. Therefore, the interface equilibrium is required to be upheld with sufficient accuracy in order to provide accurate results. In order to determine the generalized stress intensity factor (GSIF) in contact problems, the alternative approach of the contour integral formulation may also be adopted. A non-singular mode of the eigen-equation produces the auxiliary mode which eliminates the line integral on free boundaries of the closed contour around the vertex point. The normal and tangential contact constraints are enforced by the penalty method. Based on the current contact state of the slave corner node, the sticking or sliding mode of enrichment is adopted. The projection point of the node on the master surface is determined in each increment and associated master and slave elements are enriched topologically or geometrically [16]. The master interface is smoothed very close to the discretized interface by the Bezier polynomials, which increases the stability of the penalty method in large sliding frictional contact problems [40]. Bouncing from free to slip condition is prevented by conventional contact mechanics algorithms and the starting contact condition remains stick [40]. Since the penalty stiffness contribution significantly changes the conditioning of the global stiffness matrix, in order to prevent possible jumps from a slip state to free state of the slave nodes, especially at a corner node (which its location describes the origin of high gradient stress field), an automatic sub-incrementation scheme, which divides the load increment into substeps that satisfy an error control criterion, is applied, as proposed by Sheng and Sloan [41] and Sheng et al. [42].

2. Asymptotic expansion in 2D linear elasticity

The asymptotic expansion method, pioneered by Wieghardt [28] and followed by Williams [5] who extended the approach to the analysis of stress singularity around the re-entrant corners in tensile plates and other singular wedge problems, expresses the self-similar (separated-variable form) solution space around a vertex point of a composite wedge in the literature for slipping and sticking punch indentation and bonded sides corner penetration in dissimilar materials [39]. In the next section, node-to-segment (NTS) slide line formulation will be enriched to reproduce the superior modes of stress singularity. The strength of singularity and the relative asymptotic parameters are determined in terms of the specified master and slave material elastic modules, corner angles and coefficients of friction. In contrast to Giner et al. [16], who studied the same case of sliding punch, the number of additional degrees of freedom is reduced as a result of combining the basic asymptotic modes in order to satisfy the characteristic equation by determining the field parameters in the pre-processing phase. The enrichment functions derived from the biharmonic equation of elasticity are the solution to several asymptotic problems with different boundary conditions, but with the same eigen-value. Therefore, the interface equilibrium is required to be upheld with sufficient accuracy in order to provide accurate results. In order to determine the generalized stress intensity factor (GSIF) in contact problems, the alternative approach of the contour integral formulation may also be adopted. A non-singular mode of the eigen-equation produces the auxiliary mode which eliminates the line integral on free boundaries of the closed contour around the vertex point. The normal and tangential contact constraints are enforced by the penalty method. Based on the current contact state of the slave corner node, the sticking or sliding mode of enrichment is adopted. The projection point of the node on the master surface is determined in each increment and associated master and slave elements are enriched topologically or geometrically [16]. The master interface is smoothed very close to the discretized interface by the Bezier polynomials, which increases the stability of the penalty method in large sliding frictional contact problems [40]. Bouncing from free to slip condition is prevented by conventional contact mechanics algorithms and the starting contact condition remains stick [40]. Since the penalty stiffness contribution significantly changes the conditioning of the global stiffness matrix, in order to prevent possible jumps from a slip state to free state of the slave nodes, especially at a corner node (which its location describes the origin of high gradient stress field), an automatic sub-incrementation scheme, which divides the load increment into substeps that satisfy an error control criterion, is applied, as proposed by Sheng and Sloan [41] and Sheng et al. [42].

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The corresponding stress and displacement fields can be written as,
\[
\hat{F}(\lambda, \theta) = \begin{bmatrix}
\sin(\lambda - 1)\theta & \cos(\lambda - 1)\theta \\
\sin(\lambda + 1)\theta & \cos(\lambda + 1)\theta
\end{bmatrix}
\]

where \( \Sigma = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} \\ \sigma_{r\theta} & \sigma_{\theta\theta} \end{bmatrix} \) and \( U = \begin{bmatrix} u_r \\ u_\theta \end{bmatrix} \) are the stress and displacement vectors in polar coordinate system, respectively; the arrays of \( \{A_j, B_j, C_j, D_j\} \) are arbitrary constants and \( \mu_j, \kappa_j \) are shear modulus and Kolosov’s constant, respectively. Substitution of these equations into the appropriate homogeneous interface boundary conditions will create four homogeneous linear constraints for wedge interfaces and two traction-free constraints for free boundaries (if exist). As a result, there exist 4n equations for 4n unknowns \( \{A_j, B_j, C_j, D_j\}_{j=1,n} \) of n wedges. The resulting algebraic eigen-equation (i.e. the characteristic equation) has several non-trivial solutions for the eigen-values \( \lambda \) and the corresponding eigen-functions in a separated variable form of \( f_j(r)g_i(\theta) \) (the \( i \)th eigen-function for the \( j \)th wedge). They are obtained by setting the determinant of the coefficients of the algebraic equation equal to zero \([16]\). The characteristic equation is a function of \( \lambda \), the elastic constants such as \( \mu_j, \kappa_j \) and the interface constants such as the Coulomb’s coefficient of friction for sliding/slipping boundaries.

The general solution in the immediate vicinity of the vertex can be written in the form of eigenfunction expansion

\[
u_j = \sum_{k=0}^\infty kr^k g_i(\theta).
\]

The solution associated with the smallest real part produces the highest deformation gradients (order of singularity). Furthermore, if this dominant solution \( \lambda_0 \) satisfies \( \text{Re}(\lambda_0) < 1 \), a singular stress field is produced. However, solutions with \( \text{Re}(\lambda_0) < 0 \) are not feasible since they lead to an infinite displacement field. Thus, only solutions which meet \( 0 < \text{Re}(\lambda_0) < 1 \) are of practical interest \([3]\).

To obtain the characteristic equation, for example for indentation of a sticking deformable punch into a semi-infinite elastic medium, the stick boundary condition is satisfied at \( \theta = 0 \) and the traction-free boundary condition is considered at \( \theta = \phi \) and \( \theta = -180^\circ \) (Fig. 2).

The characteristic equation is treated via the standard Newton’s method. The order of the stress singularity versus the corner angle \( \phi \) is depicted in Fig. 3. In this figure, the Young’s modulus and Poisson’s ratio of steel and concrete are \( E\text{steel} = 200 \text{ GPa}; \nu\text{steel} = 0.30 \) and \( E\text{conc} = 20 \text{ GPa}; \nu\text{conc} = 0.25 \).

As shown in Fig. 2, a couple of singular eigen-modes are probable to be reproduced for intermediate punch angles \( \phi \). The order of singularity is \( \lambda = 1 \) (non-singular) for \( \phi = 0 \) and \( \lambda = 0.5 \) for \( \phi = 180^\circ \) irrespective of the material dissimilarity. The results of analytical solution shows that for \( 0 < \phi < 180^\circ \), the solution curve has two branches for the real part of singularity order. Two threshold angles (key angles) may be distinguished for every material composition of slave and master media (see Fig. 3). The first key angle at A introduces the entrance of the second singular mode and the second key angle represents the point in which two eigenvalues approach to each other. Before the first key angle, the sticking punch configuration has only one singular mode and after the second key point at B (\( \phi = 90^\circ \) for steel/concrete contact pair), the two eigenvalues are complex conjugates of each other. For similar materials, such as the steel/steel in Fig. 3, this second key angle occurs at 180° with the well-known square root singularity (i.e. \( \lambda_1 = \lambda_2 = 0.5 \)).

Having obtained the eigenvalue \( \lambda \), the parameters of the mode-shapes of the asymptotic expansion around the vertex point \( \{A_j, B_j, C_j, D_j\}_{j=1,n} \) are derived by the conventional Gauss reduction and back-substitution processes on the characteristic equation. The following functions are then derived for computing the analytical displacement field in the Cartesian coordinates,

\[
[u]_i = [d]_i[F]_j (i = x, y; j = 1, 4)
\]

Fig. 2. Configuration of a complete sliding contact.
\[ [F] = r^2 \begin{bmatrix} \sin(\lambda - 2) \theta & \sin \lambda \theta & \cos(\lambda - 2) \theta & \cos \lambda \theta \end{bmatrix} \]  
\[ [d] = \frac{K_{CST}}{\mu} \begin{bmatrix} -iD & kD - (\lambda + 1)C & -iB & kB - (\lambda + 1)A \\ iB & kB + (\lambda + 1)A & -iD & -kD - (\lambda + 1)C \end{bmatrix} \]

where \( u_i \) is the displacement field in the local coordinate system, \( \mu \) is the shear modulus of the wedge and \( K_{CST} \) is the generalized stress intensity factor. Eq. (5) will be used to define a consistent enrichment function to be embedded within the PUFEM.

### 3. Sliding punch corner indentation

Gdoutos and Theocaris [29] and Comninou [30] have derived the eigen-equation for a complete sliding punch on a dissimilar plane surface, as described in Fig. 2.

\[ \Delta(\lambda; \phi, \alpha, \beta, \mu) = 8(1 + \lambda) \sin \lambda \pi \times \left[ (1 + \lambda) \cos \lambda \pi(\sin^2 \lambda \phi - \lambda^2 \sin^2 \phi) + \frac{1}{2}(1 - \alpha) \sin \lambda \pi(\sin 2 \lambda \phi + \lambda \sin 2 \phi) \right. \]

\[ \left. + \mu \sin \lambda \pi \left( (1 - \alpha) \lambda(1 + \lambda) \sin^2 \phi - 2\beta(\sin^2 \lambda \phi - \lambda^2 \sin^2 \phi) \right) \right] = 0 \]

where \( \phi \) is the wedge angle and \( \alpha = \frac{(\mu/\nu)/(\kappa + 1) - (\kappa - 1)}{(\mu/\nu)/(\kappa + 1) + (\kappa - 1)} \) and \( \beta = \frac{(\mu/\nu)/(\kappa + 1) - (\kappa - 1)}{(\mu/\nu)/(\kappa + 1) + (\kappa - 1)} \) are the Dundurs parameters, \( \mu_i \) is the coefficient of friction and \( \kappa = 3 - 4v \) for plane strain problems. The characteristic Eq. (8) has been constructed from the asymptotic expansion of the following equations:

\[ \sigma_{xy}^{(1)} + \mu \sigma_{yy}^{(1)} = 0; \sigma_{xx}^{(1)} = \sigma_{xy}^{(2)} = \sigma_{yy}^{(2)} = \sigma_{zz}^{(2)} = u_{iy}^{(2)} \text{ at } \theta = 0 \]

\[ \sigma_{xy}^{(1)} = 0 \text{ at } \theta = \phi; \sigma_{xy}^{(2)} = 0 \text{ at } \theta = -\pi \]

Eq. (8) may be solved numerically by using a mathematical software such as Maple, Mathematica or Matlab which support symbolic equation manipulations or parametric solution. The result of the rather tedious symbolic calculations on the characteristic equation will be the real and imaginary parts of the characteristic determinant and their parametric derivatives with respect to real and imaginary parts of the eigenvalue \( \lambda \). Here, however, the Newton’s iterative method is adopted to solve for the dominant \( \lambda \),

\[ \lambda = \varepsilon + i\eta \]

and the characteristic equation, typically written in terms of \( \Delta(\lambda) \), is decomposed into its real and imaginary parts,

\[ \Delta(\lambda) = \Delta(\varepsilon, \eta) + i \Delta(\varepsilon, \eta) = 0 \]

Then, two sequential iterations for \( \varepsilon \) and \( \eta \) are to be performed to converge to exact values within the predefined error tolerance and the maximum number of iterations,

\[ e_{i+1} = e_i - d\varepsilon; d\varepsilon = - \frac{\Delta \varepsilon \partial \Delta / \partial \varepsilon}{\partial \Delta / \partial \varepsilon \partial \Delta / \partial \eta}; d\eta = - \frac{\Delta \eta \partial \Delta / \partial \eta}{\partial \Delta / \partial \varepsilon \partial \Delta / \partial \eta}; \eta_{i+1} = \eta_i - d\eta; d\varepsilon = - \frac{\Delta \varepsilon \partial \Delta / \partial \varepsilon}{\partial \Delta / \partial \varepsilon \partial \Delta / \partial \eta} \]

\[ d\eta = - \frac{\Delta \eta \partial \Delta / \partial \eta}{\partial \Delta / \partial \varepsilon \partial \Delta / \partial \eta} \]
The components $\frac{\partial}{\partial \epsilon^k}(k, l = 1, 2)$ are determined explicitly in terms of $\epsilon$ and $\eta$. Unnecessary parametric calculations can be avoided, knowing that: $\frac{\partial}{\partial \epsilon^k} \Delta_j = -\frac{\partial}{\partial \eta^j} \Delta_j \frac{\partial}{\partial \epsilon^l} \Delta_j$. Setting $D = 1$ (the last array of the eigenvector), the back-substitution process will extract the other arrays of the eigenvector.

The solution of the eigen-equation for plane strain sliding of a steel punch on different substrates is shown in Fig. 4 which demonstrates the trend of variation of the singularity power with respect to the vertex angle for various friction coefficients. The elasticity parameters for steel are $E_s = 200$ GPa; $G_v = 0.3$.

The order of stress singularity at the vertex depends on the direction in which the wedge slips with respect to the half space. In Fig. 4, positive coefficients of friction indicate sliding towards the vertex point (leading corner) and sliding away from the corner (trailing corner) produces lower singularity order. Such a regime governs stiffer slave and softer substrates only for acute angles. For angles greater than $90^\circ$, the trend reverses and as shown in Fig. 4, for a right angle corner of a steel specimen sliding on the substrate 3, the order of singularity is independent of the coefficient of friction. The corner angle at which the singularity power becomes independent of the coefficient of friction varies for different punch/substrate stiffness ratios. For example, for a substrate of elastic parameters $E_c = 10$ GPa; $v_r = 0.25$ (substrate 4), the singularity power for steel corner of $69.3^\circ$ becomes independent of the coefficient of friction and is equal to $k = 0.5841$. While for the substrate 2 with parameters $E_c = 30$ GPa; $v_r = 0.25$, such a friction independent punch angle is $102.26^\circ$ with the corresponding order of singularity $k = 0.5411$.

The variation of singularity power with respect to the coefficient of friction and sliding direction for $60^\circ$, $90^\circ$ and $120^\circ$ angles is depicted in Fig. 5. Again, negative friction coefficient corresponds to the sliding away from the corner point. The order of stress singularity increases by the increase of the coefficient of friction for relatively stiff punch with corner angles lower than $90^\circ$. For obtuse angles the trend is reversed with a mild rate. The variation of the order of singularity with respect

![Fig. 4. Singularity power of a sliding steel punch corner versus corner angle. Upper left: Steel substrate 1 upper right: Substrate 2, $E_s = 30$ GPa, $v_c = 0.25$. Lower left: Substrate 3, $E_s = 20$ GPa, $v_c = 0.25$. Lower right: Substrate 4, $E_s = 10$ GPa, $v_c = 0.25.


to the relative elastic parameters is described in Fig. 6. It can be concluded from the results that for all trailing corners and the leading corner with relatively low friction coefficients, the order of singularity reduces for indenting low stiffness wedges. Similar conclusions were reported by Dundurs and Lee [43]. Dundurs and Lee [24,25] have also stated that the characteristic equation has just one real root [43]. The present solution of the roots of \( \Delta(\lambda; \phi, \alpha, \beta, \mu) \) (Eq. (8)), confirms that there is at most one real root in the interval \( 0 < \lambda < 1 \) [44,45].

Logarithmic singularities accommodate the region between no singularity and power singularities (Williams-type singularities) at transition threshold \( \lambda = 1 \), where the singularity vanishes for \( \lambda > 1 \), while \( \lambda < 1 \) cases represent an exponential singularity. As discussed by Dundurs and Lee [43], a logarithmic solution stems from the double roots of the basic characteristic equation of the biharmonic equation. Supposing the separated variable exponential form of the Airy stress function as \( \phi = Qe^{(\lambda \pm 1)\ln r}g(\theta) \), the solution for angular function \( g(\theta) \) leads to \( g(\theta) = e^{\omega \theta} \) where \( \omega \) is the solution of the following characteristic equation,

\[
\omega^2 + 2(\lambda^2 + 1)\omega^2 + (\lambda^2 - 1)^2 = 0
\]  

(14)

The solutions of (14) are explicitly derived as \( \omega = \mp i(\lambda \pm 1) \). Existence of the identical solutions \( \omega_0 = \omega_0h \), which occurs when \( \lambda \in \{-1, 0, 1\} \), implies that the derivative of the angular mode with respect to eigenvalue, \( \frac{\partial g(\theta)}{\partial \omega} \), to be the other conjugate pair according to the method of Frobenius for multiple roots of the characteristic equation. In a similar manner for the radial part of the Airy stress function \( f(\lambda) = r^\lambda \), when the eigenvalues corresponding to conjugate eigenfunctions come close to each other on limit \( (\lambda_1 = \lambda_2) \), the mixed mode case of double roots incorporates a logarithmic partner as \( \frac{\partial f(\lambda)}{\partial \lambda} = r^\lambda \ln r \). Dempsey and Sinclair were the first to obtain the logarithmic results from the derivative of the Williams solution with respect to the eigenvalue. The non-separated form of the solution is justified as [12]:

\[
\phi(\rho, \theta) = \phi_1 + \phi_2 = K_1 r^\lambda g_1(\theta) + K_2 r^{\lambda + 1} g_2(\theta)
\]

\[
\lim_{\lambda \to \lambda_1} \phi_1 = \phi_2 = \phi_1 + \frac{\partial \phi_1}{\partial \lambda} \Delta \lambda
\]

\[
\Rightarrow \phi(\rho, \theta) = K_1 r^\lambda g_1(\theta) + \overline{K}_2 r^{\lambda + 1} \left[ (\ln r) g_1(\theta) + \frac{\partial g_1(\theta)}{\partial \lambda} \right]
\]

(15)

After substituting the angular modes of (1) for \( g_1(\theta) \) in (15), the general form of the solution for double root solutions of Eq. (14) is expressed as:

\[
\phi_0(\rho, \theta) = \overline{K}_1 r^\lambda \left\{ A \sin(\lambda + 1)\theta + B \cos(\lambda + 1)\theta + C \sin(\lambda - 1)\theta + D \cos(\lambda - 1)\theta \right\} \\
+ \overline{K}_2 r^{\lambda + 1} \left\{ A' \sin(\lambda + 1)\theta + B' \cos(\lambda + 1)\theta + C' \sin(\lambda - 1)\theta + D' \cos(\lambda - 1)\theta \right\} \\
+ \theta \cos(\lambda + 1)\theta + B[\ln r] \cos(\lambda + 1)\theta + \theta \sin(\lambda + 1)\theta + C[\ln r] \sin(\lambda - 1)\theta + \theta \cos(\lambda - 1)\theta \\
+ D[\ln r] \cos(\lambda - 1)\theta + \theta \sin(\lambda - 1)\theta \}
\]

(16)

For the special case of \( \lambda = 1 \), which is always among the eigenvalues of the asymptotic characteristic equation, there is a logarithmic (non-separated) Airy stress function expressed as [12],

\[
\phi_0(\rho, \theta) = \overline{K}_1 r^\lambda \left\{ A + B\theta + C \sin 2\theta + D \cos 2\theta \right\} \\
+ \overline{K}_2 r^{\lambda + 1} \left\{ A' + B'\theta + C' \sin 2\theta + D' \cos 2\theta \right\} + (\ln r) \left[ A + B\theta + C \sin 2\theta + D \cos 2\theta \right] + C0 \cos 2\theta - D0 \sin 2\theta
\]

(17)
Consequently, similar to discussions of Dundurs and Lee [43] for the case of a sliding punch, $k = 1$ pertains to the logarithmic singularity and this non-separable form of singularity is possible for all wedge angles. This is an expected result since the normal traction $r_{hh}$ is not likely to vanish at the vertex of the wedge ($x = 0^+$) even for small wedge angles. Thus, shearing traction $r_{r}$ under slip criterion undergoes a jump from $x = 0^+$ to $x = 0^-$. It is known, however, that in the present case a jump in shearing traction $r_{rr}$ leads to a logarithmic singularity in the tangential component of normal stress (i.e. $\sigma_{rr}$) [43,44]. It should be noted that, the logarithmic singularity does not govern the solution in the presence of a unique real order of singularity and thus it is not of major concern here.

In order to implement the discussed procedure, the node to segment (NTS) slideline frictional contact formulation is adopted. A smooth $C^1$ discretization of the master surface via the Bezier interpolants (see Fig. 7) is used to reduce the oscillations and instabilities due to jump in surface normal direction on the discretized contact interfaces, where lower order finite element edges may create artificial corners other than the target singular corner [40].

In order to control the effect of penalty contribution on global stiffness matrix, which may become ill-conditioned due to the significant difference between the slave and master modules (e.g. a steel punch on a soft soil), an automatic load stepping scheme in each load increment, as proposed by Sheng et al. [42], is adopted to control the rate of convergence. The first-order Euler (Newton–Raphson linearization), Eq. (18), and the second-order modified Euler update, Eq. (19), are used to construct a measure of the load path error within the loop of each increment [42],

$$\Delta U_1 = [K(U_{n-1})]^{-1}(F_{n}^{\text{ext}} - F_{n-1}^{\text{ext}})$$  \hspace{1cm} (18)

$$\Delta U_2 = [K(U_{n-1} + \Delta U_1)]^{-1}(F_{n}^{\text{ext}} - F_{n-1}^{\text{ext}})$$  \hspace{1cm} (19)

where $K(U_i)$ indicates the total stiffness matrix (classical plus constrained terms) due to updated configuration at the end of $i$th increment $U_i$ and $F_{n}^{\text{ext}}$ is the external load vector at the same increment. The error control criterion is then defined as [41]

$$R = \frac{1}{2} \frac{\|\Delta U_2 - \Delta U_1\|}{\|U_{n-1}\|} \leq DTOL$$  \hspace{1cm} (20)

where $DTOL$ is a user defined tolerance, set one order of magnitude greater than the tolerance of convergence of the inner loop (updated Lagrangian iterations). The next trial load step size of the sub-increment loop $\Delta T_{n+1}$ may be defined based on the current converged load step size,
\[
\Delta T_{p+1} = \sqrt{\frac{DTOL}{R}} \Delta T_p
\]

Equilibrium is achieved during each subincrement via the well-known updated Lagrangian formulation.

4. Partition of unity finite element method

In order to efficiently incorporate the singular sliding corner stress fields within the standard displacement-based finite element approximation, a partition of unity (PU) framework has been used. In contrast to the similar procedure by Giner et al. [16], the asymptotic field which satisfies the expected interface boundary condition on the enriched element boundaries is characterized in the pre-processing step and applied topologically (only over elements that are connected to the singular vertex slave node or its projection point) or geometrically (over elements within a process zone around the singular point in slave and master media) by 1 additional (enriched) function per local coordinate of the enriched elements (compared with four additional DOFs in [16]). Since the shape functions are compatible with the local boundary condition, the rate of convergence is increased compared with the approach proposed by Giner et al. [16]. Furthermore, as the analytical expression of the asymptotic displacement is accomplished numerically in the pre-processing stage, the application of PU-FEM analysis of sharp indentation contact problems may well be extended to other singular cases which no explicit expression is available for the characteristic equation, such as a sliding cone penetration.

The partition of unity finite element approximation is written as:

\[
u^p(x) = \sum_{i \in \text{nodes}} N_i(x)u_i + \sum_{j \in \text{enr. nodes}} N_j(x) \sum_{n=1,n_2} \left[ F_n(x) - F_n(x_j) \right] a_{j, n}
\]

where \(u_i\) is the displacement at node \(i\) and \(N_i(x)\) is the corresponding shape function. The second term in Eq. (22) is the extrinsic enrichment part of PUFEM which allows for implementation of higher order asymptotic functions locally in the vicinity of the sharp contact corner [37,45,46], \(F_n(x) - F_n(x_j)\), \(n_2\) are the set of enrichment functions shifted by the nodal value of the same function on each degree of freedom to ensure interpolation property of the whole approximation and \(a_{j, n}\) are the additional degrees of freedom associated with the enrichment functions \(F_n(x)\) and compact support of \(N_j(x)\). Having preserved the advantages of finite element method, such as symmetry, sparsity of the stiffness matrix and essential degrees of consistency of the approximation to solve certain differential equations, the enriched formulation has now the ability to represent a singular field of arbitrary order at any location within an element due to the partition of unity property [46].

Here, the enrichment functions are the first term of asymptotic expansion for \(u(x)\) and \(t(x)\) in local \(x\) and \(y\) coordinates of the singular point, respectively,

\[
[F(x); \tilde{F}(x)] = d_{ij} F_j(x) (i = x, y; j = 1, 4)
\]

where \(d_{ij}\) and \(F_j(x)\) are defined in (6) and (7). A typical surface plot for the enriched trial spaces \(\{N_i, F(x); N_i, \tilde{F}(x)\}\) is depicted in Fig. 8. The mode shape corresponds to sliding of a leading right angle steel punch over a steel substrate with coefficient of friction \(\mu_2 = 1.6\).

As discussed in Giner et al. [16] and many previous studies in XFEM and PUFEM, enrichments may be implemented on the nodes of the neighboring elements, known as the topological enrichment, or over a fixed influence domain, called geometric enrichment. Each enriched node will then have two additional degrees of freedom which approximate the generalized stress intensity factor at that node. The strain–displacement matrix for enriched degrees of freedom is defined as:

![Fig. 8. Surface plot of \(N_i, F(x)\) (Left) and \(N_i, \tilde{F}(x)\) (Right) located on the vertex point of the sliding indenter.](image-url)
\[
\bar{e}(X) = B(X)\alpha; B(X) = \begin{bmatrix}
\frac{\partial(N\bar{F})}{\partial x} & 0 & \frac{\partial(N\bar{F})}{\partial y} \\
0 & \frac{\partial(N\bar{F})}{\partial y} & \frac{\partial(N\bar{F})}{\partial x} \\
-\frac{\partial(N\bar{F})}{\partial y} & -\frac{\partial(N\bar{F})}{\partial x} & 0
\end{bmatrix}; \alpha = \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix}
\]

where \(\bar{e}(X)\) is the strain contribution of enrichment. The order of singularity and other parameters of the enrichment function for a certain vertex singularity problem (e.g. sticking or slipping punch corner on a dissimilar substrate or sharp corner symmetric sliding, etc.) are determined numerically in advance. For example, in the benchmark problem introduced in Giner et al. [16] for sliding of a steel punch on a steel substrate, the leading/trailing corners have the eigenvalues shown in Fig. 9. In this figure, solid lines trace the location of zeros of the real part of the characteristic Eq. (8), while the dashed lines scan the location of zeros for the imaginary part. Points of intersection of the continuous and dashed lines represent a root of the characteristic Eq. (8) in \(\text{Re}(\tilde{\lambda}) - \text{Im}(\tilde{\lambda})\) plane. It is noted that the whole Fig. 9 corresponds to a single point in Fig. 4.

Similar diagrams can be generated for other interface conditions. For example, for the sliding of a 60⁰ steel punch corner on a steel substrate of various materials, trailing and leading corners create the eigenvalues demonstrated in Fig. 10. Elastic properties for steel are \(E_s = 200\) GPa, \(v_s = 0.3\) and for concrete are \(E_c = 20\) GPa, \(v_c = 0.25\).

The dominant singularity power, as shown in Figs. 9 and 10, and associated eigenvectors are derived in the preprocessing phase (before the calculation of the overall stiffness matrix and after detecting the singular vertices in the problem domain), as presented in Table 1.

In order to characterize the intensity of the singularity, a domain integral formulation can also be adopted to compute the generalized stress intensity factor (GSIF). The use of partition of unity enrichment leads to accurate estimations of GSIFs on relatively coarse meshes, which is particularly beneficial for modeling non-linear sliding contacts. Since the consistent asymptotic fields have been adopted as enrichment functions, additional unknowns of each enriched node directly represent the generalized stress intensity factors, removing the additional burden of extracting \(K_{GSIF}\). For comparison purposes, however, a contour integral is exploited to derive \(K_{GSIF}\) in the standard finite element domain where enriched degrees of freedom are not present. As explained in [16], the general method of determining this parameter in contact problems is based on the Betti’s reciprocal work theorem [47].

Consider a primary and an auxiliary state of equilibrium, \(\{\sigma_{ij}, u_i\}\) and \(\{\sigma_{ij}', u_i'\}\), respectively, applied on a simply-connected region \(\Omega\) which excludes any singular stress point (as described in Fig. 11 for a punch sliding problem), the reciprocal work theorem is expressed as [16,47],

\[
\oint_{\Gamma} t_i u_i' d\Gamma = \oint_{\Gamma} t_i' u_i d\Gamma
\]

where \(\Gamma = \Gamma_p \cup \Gamma_1 \cup \Gamma_m \cup \Gamma_s\) denotes the contour path. If the boundary tractions due to the auxiliary stress field vanish on \(\Gamma_p, \Gamma_m\), then the contribution of the reciprocal work on \(\Gamma_s + \Gamma_m\) is omitted and (25) can be re-written in the Somigliana’s identity form as,

\[
I_z = \oint_{-\Gamma_i (t=t_s)} (t_i u_i' - t_i' u_i) d\Gamma = \oint_{\Gamma_1} (t_i u_i' - t_i' u_i) d\Gamma
\]

Fig. 9. Roots of the real and imaginary parts (solid and dash lines, respectively) of the characteristic Eq. (8) of the sliding punch problem \(\phi = 90^\circ, \mu_s = 1.6\), Left: Trailing corner and Right: Leading corner.
For this purpose, the auxiliary field should be a non-singular asymptotic field. In Eqs. (25) and (26), \{t, u\} and \{t/C3, u/C3\} are traction and displacement vectors corresponding to conjugate asymptotic fields. For example, the auxiliary field corresponding to dominant modes of Table 1 may be considered according to Table 2. In Eq. (26), it is assumed that the asymptotic solution is best reproduced at radius \(r_e\). Writing \{t, u\} on the left hand side of (26) in terms of the analytical asymptotic solution and \{t, u\} on the right hand side in terms of the finite element approximation, \(K_{GSIF}\) can be estimated as:

\[
K_{GSIF} = \frac{1}{2} \left( \frac{t}{u} \right)_{LH} - \frac{1}{2} \left( \frac{t}{u} \right)_{RH}
\]

For this purpose, the auxiliary field should be a non-singular asymptotic field. In Eqs. (25) and (26), \{t, u\} and \{t', u'\} are traction and displacement vectors corresponding to conjugate asymptotic fields. For example, the auxiliary field corresponding to dominant modes of Table 1 may be considered according to Table 2. In Eq. (26), it is assumed that the asymptotic solution is best reproduced at radius \(r_e\). Writing \{t, u\} on the left hand side of (26) in terms of the analytical asymptotic solution and \{t, u\} on the right hand side in terms of the finite element approximation, \(K_{GSIF}\) can be estimated as:

\[
K_{GSIF} = \frac{1}{2} \left( \frac{t}{u} \right)_{LH} - \frac{1}{2} \left( \frac{t}{u} \right)_{RH}
\]

### Table 1

<table>
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<tr>
<th>Material</th>
<th>Corner angle</th>
<th>(\mu_f)</th>
<th>(\lambda)</th>
<th>(A_1)</th>
<th>(B_1)</th>
<th>(C_1)</th>
<th>(D_1)</th>
<th>(A_2)</th>
<th>(B_2)</th>
<th>(C_2)</th>
<th>(D_2)</th>
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<td>Steel/Steel</td>
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<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
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<td>-0.157</td>
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<tr>
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<td>1.000</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
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<tr>
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### Table 2

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<th>(\lambda)</th>
<th>(\lambda')</th>
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<th>(B_1)</th>
<th>(C_1)</th>
<th>(D_1)</th>
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<th>(B_2)</th>
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<tbody>
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<tr>
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<td>0.25</td>
<td>1.00</td>
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Fig. 10. Roots of the real and imaginary parts (solid and dash lines, respectively) of the characteristic Eq. (8), plane strain steel punch/concrete substrate \(\phi = 60^\circ; \mu_f = 1\). Left: Trailing corner and Right: Leading corner.

Fig. 11. Contour integral for determining the generalized stress intensity factor (GSIF).
\[ K_{GSIF} = \frac{1}{C_{l,x}} \int_{\Gamma_1} (t_i u_i' - t'_i u_i) d\Gamma \]  
(27)  
\[ C_{l,x} = \frac{1}{2} \int_{\Omega_{EDL}} \left[ \frac{\partial r_{ij} u_j'}{\partial x} + \frac{\partial r_{ij} u_j'}{\partial y} - \frac{\partial r_{ij} u_j'}{\partial y} - \frac{\partial r_{ij} u_j'}{\partial x} \right] d\Omega \]  
(28)

where \( \{r_{ij}, u_j\} \) and \( \{\tilde{r}_{ij}, \tilde{u}_j\} \) are the angular terms of the expansion given in (2) and (3) for dominant and auxiliary modes, respectively. It should be mentioned that the rigid body motion of the reference location and the initial stresses which are independent of the relative displacement field shall be eliminated from the numerical results of \( \{\sigma_{ii}, u_i\} \) in evaluating the right hand side integral of (27). The equivalent domain integral form of (27) is developed via introducing a scalar function \( q(X) \) which is equal to unity on the inner boundary of a band of elements around the vertex point \( \Omega_{EDL} \) zero on the outer boundary of the band and smooth across the band [47]. Using the divergence theorem, the contour integral is converted into a domain form as:

\[ l_z = \int_{\Omega_{dis}} [\sigma_{ij} u_j' - \sigma_{ij}' (u_i - u_{0i})] q d\Omega + \int_{\Omega_{int}} [\sigma_{ij} u_j' - \sigma_{ij}' (u_i - u_{0i})] q_j d\Omega \]  
(29)

where \( u_{0i} \) is the displacement vector of the vertex point and \( \Omega_{EDL} \) is the equivalent domain integral region. Owing to the fact that the body force is zero and the body is in equilibrium state \( (\sigma_{ij} = \sigma_{ij}' = 0) \), the first integral of \( l_z (29) \) vanishes and \( K_{GSIF} \) can be computed as:

\[ K_{GSIF} = \frac{1}{C_{l,x}} \int_{\Omega_{dis}} \left[ \sigma_{ij} u_j' - \sigma_{ij}' (u_i - u_{0i}) \right] q_j d\Omega \]  
(30)

5. Numerical examples

In order to assess the accuracy of the proposed contact formulation, a simple test is performed to determine the coefficient of friction \( \mu_f \) which prevents a block from sliding over an inclined substrate of angle \( \theta \). A fully implicit Newmark time integration scheme is used to solve the dynamic equilibrium. The time step size is 1 s and the sub-increments of the time steps are fulfilled by the same error control algorithm. The analytical solution is \( \mu_f = \tan \theta = 0.5 \). The configuration of the problem is shown in Fig. 12.

The material is assumed to be steel with density of \( 7.8 \times 10^3 \) kg/m\(^3\) and the penalty factor for normal and tangential contacts is \( 2.0 \times 10^3 \) kN/m. Fig. 13 demonstrates the sliding sequence for the coefficient of friction of \( \mu_f = 0.505 \) which is obtained by setting \( DTOL = 1.0 \times 10^{-2} \), and the tolerance of convergence of dynamic residual force is \( ITOL = 1.0 \times 10^{-4} \). The total duration is 12 s and the total number of iterations and substeps are 7726 and 262, respectively. For \( \mu_f = 0.495 \) the stages of sliding for three different times are depicted in Fig. 14. Also, the stress and strain contours for the sliding block with \( \mu_f = 0.505 \) for the first increment are shown in Fig. 15.

In addition, the contact state of each slave node and associated normal and tangential forces are illustrated in Fig. 16. The tangential force is compared with the limit tangential force due to Coulomb’s slip criterion for \( \mu_f = 0.495 \) and \( \mu_f = 0.505 \) cases.

The external force increment for the error control algorithm is the explicit residual force vector, as given in (20). The first and second order displacement increments for the dynamic procedure differ from that defined in (18), (19) and are specified as.

![Fig. 12. Configuration of the benchmark problem.](image-url)
Fig. 13. Stages of sliding of block for $\mu_f = 0.505$. Left: $t = 1$ s, Middle: $t = 8$ s, Right: $t = 11$ s.

Fig. 14. Stages of sliding of block for $\mu_f = 0.495$. Left: $t = 1$ s, Middle: $t = 8$ s, Right: $t = 11$ s.

Fig. 15. Contour of stresses at $t = 1$ s. Left: $\sigma_{xx}$, Middle: $\sigma_{yy}$, Right: $\sigma_{xy}$.

Fig. 16. Normal and tangential forces at $t = 1$ s. Left: $\mu_f = 0.405$ and Right: $\mu_f = 0.505$. 
\[ \Delta U_1 = \Delta T_n v_n + \frac{\Delta T^2}{2} (1 - 2\beta)a_n \]  
\[ \Delta U_2 = \left[ K^*(\tilde{U}_{n+1}) \right]^{-1} R(\tilde{U}_{n+1}) \]  

where \( K^*(\tilde{U}_{n+1}) \) and \( R(\tilde{U}_{n+1}) \) are the equivalent stiffness matrix and residual vector for the predicted configuration at time \( T_{n+1} \), respectively; \( \Delta T_n \), \( v_n \), and \( a_n \) are the time increment, velocity, and acceleration vectors at time \( T_n \) and \( \beta \) is the Newmark’s parameter. It should be mentioned that, the rate of sliding is influenced by the increase of the coefficient of friction. Since the treatment of contact constraint by the penalty method (i.e. stiffness approach of constraint enforcement) in a dynamic process does not necessarily satisfy the continuity of accelerations on the interface, the need for both interfaces to reach to a kinematic equilibrium state in an explicit or implicit time integration scheme requires a more efficient time stepping criterion to maintain the continuity of velocity/acceleration over the interface in the absence of an additional constraint which satisfies the equality of the dynamic parameters of slave and master conjugates. Any inaccuracy in local inertia forces in normal direction to the contact interface affects the magnitude and distribution of the normal contact tractions and consequently, the slip criterion and contact state of the slave nodes are influenced during the subsequent increments.

The second benchmark problem assesses the superiority of PUFEM upon the standard finite element method. Two advantages are sought by enriching the local region of singularity in contact problems. First to obtain better estimates of GSIF which is useful for design purposes. Secondly, to more efficiently approximate higher order fields with a reasonably coarse mesh. Analytical displacement and tractions are imposed on the far field regions from the vertex point. The GSIF contours are presented for different radius of integral band. It should be noted that, for the enriched case, the GSIF parameter is directly extracted via additional unknowns contributed to the enrichment function on the vertex node (called the direct method in this paper). Configuration of the problem is defined in Fig. 17. The mesh consists of 1316 nodes and 1220 elements (mesh 1.5 x 3.0, with minimum and maximum element lengths of 1.5 and 3.0 on interface CD and far boundaries AB and GH, respectively). Squared nodes of the interface and upper and bottom boundaries of the slave and master blocks are prescribed by asymptotic displacement field with \( K_{GSIF} = -0.1 \). On the other boundaries, asymptotic tractions are imposed. Asterisk nodes near the corner vertex node describe the topological enriched nodes.

For the topological enrichment, only three elements of the mesh are enriched (1 slave and 2 target elements), whereas for geometric enrichment, all elements within a fixed radius of enrichment of \( r_e = 3 \text{ cm} \) are enriched (comprising 5 elements; 1 slave and 4 target elements). The overall view of the shear stress \( \sigma_{xy} \) contour and its distribution over the singular slave element (see Fig. 17) for the standard FE and geometrically enriched PUFE cases are shown in Fig. 18. It is obvious that the PUFEM solution represents the exact localized field, whereas in the standard FE case a smooth stress field is generated, which overestimates the exact solution in regions far away from the singular point (D) and underestimates that significantly around the singular point. Clearly, the enriched cases are capable of describing the singularity in contrast to the standard FEM. The local distribution of the reproduced singular field in the enriched element is visualized in Fig. 19, where a local region of dimensions 0.1% of the interface width has been selected to illustrate the local stress field.

To examine the effect of discretization, the generalized stress intensity factor is derived with both approaches for different meshes of the master and slave bodies, as typically depicted in Fig. 20. The order of Gauss points for the enriched and blending elements (elements which are partially enriched) is 2 x 2 with 25 x 25 sub-quads and 2 x 2 for the standard finite elements. Results of GSIF, obtained from the contour integral approach for FEM and PUFEM with different enrichment strat-
The results indicate that the direct method which reproduces the GSIF explicitly on the additional (enrichment) degrees of freedom situated on the vertex node has substantially improved the accuracy of PUFEM approach. This can be attributed to the implementation of the complete (consistent) asymptotic fields of the mode of singularity.

It is clear that, the GSIF results of contour integral approach is enhanced by using geometrical enrichment and the accuracy is one order of magnitude higher than that obtained from the standard finite element method. However, due to discretization error in regions around the singular node in FE case and the assumptions made in the equivalent domain integral form of (27), as discussed in [16], the accuracy may substantially be reduced for contour paths close to the singular nodes. On the other hand, as will be discussed later, due to severely localized nature of the sliding punch corner singularity, asymptotic fields may not always be reproduced accurately, even though the enrichment is applied beyond the contour path. Moreover, the domain of contour integral has a gap (discontinuity) along the contact interface and the two discrete media may undergo a rigid rotation with respect to the vertex point, which can be regarded as another source of error in calculating the numerical integral around the singular point. Therefore, the contour integral approach with formulation introduced in (27)–(29) is not suitable for general contact indentation problems concerned in this paper. Meanwhile, the proposed direct method, not only eliminates the post processing indentation costs of the conventional approach, but also puts forward a better accuracy with the same rate of convergence (as discussed below) for an actual contact problem.
As mentioned before, the most important advantage of the local enrichment of finite element interpolation space is to achieve a more precise estimation of the GSIF parameter. Enrichment gives the opportunity to produce the expected higher order field locally by a relatively coarse mesh. However, in contrast to the crack-tip asymptotic field which is controlled by the local boundary conditions along one direction, the boundary conditions along at least two directions are effective in reproducing the asymptotic fields in a punch-substrate sliding corner. For this reason, the process zone of the stress singularity in contact corners has a limited length compared with the crack problem. Enrichment functions of the crack problem or any singular contact configuration participate in the finite element solution due to satisfaction of their respective boundary conditions. Consistent asymptotic fields, the finite element mesh and other implicit enrichment functions introduce the boundary conditions that should be compatible with each other to generate the expected solution. On one hand, the displacement-based finite element method has negligible control on the stress state, specially to satisfy boundary conditions such as \( \sigma_{11}^{(1)} + \mu \sigma_{22}^{(1)} = 0 \) with the NTS penalty method, while on the other hand, other interface conditions i.e. \( \sigma_{11}^{(2)} \sigma_{22}^{(2)} \sigma_{33}^{(2)} u_1^{(2)} = u_2^{(2)} \) which describe the contact interaction of the slave and target bodies, necessitate an implicit contact interface definition or at least, a high resolution of the state variables on the interface when contact faces are defined explicitly via the PUFEM strategy. As a result, the enriched element, attributed to the singular vertex node, is theoretically more eligible to reproduce the asymptotic fields.
Fig. 21. Accuracy of the contour integral for various domain integral radius $r_q$ (mesh $1.5 \times 3.0$; 1316 nodes, 1220 elements).

Fig. 22. Accuracy of the direct and contour integral approaches versus enrichment radius $r_e$ (mesh $1.5 \times 3.0$; 1316 nodes, 1220 elements).

Fig. 25 compares the accuracy of GSIF of the direct method as opposed to the contour integral method for two similar meshes versus different enrichment radius (for the direct method) or contour integral band radius (for the contour integral method). It should be mentioned that due to singular/high gradient fields within the span of enrichment functions, error in additional unknowns, attributed to the enriched degrees of freedom, can disturb the solution precision. Therefore, a sufficiently higher order of integration is required in enriched elements to control the local error. Effect of the order of Gauss integration on the rate of convergence is examined in Fig. 26. Since the standard Gauss quadrature is not consistent with non-polynomial shape functions, in order to minimize the error of integration of a singular field near the slave singular node and master element boundary, all enriched elements are subdivided to quads. In contrary to sub-triangulation, sub-quads maintain the address of the initial Gauss points during the increments of a nonlinear procedure and so that the tracking of path-dependent parameters are facilitated.

The smoothed rate of mesh size convergence of the finite element and different PUFEM simulations is demonstrated in Fig. 26. A linear regression is performed to determine an equivalent convergence rate for the oscillating results, which can be attributed to the mesh sensitivity of the local solution and unstructured mesh discretization. The ratio of the element size on
the interface has been considered to be five times smaller than that on the upper and lower boundaries. It is clearly observed that the results of the direct method for geometrical enrichment is more precise with far improved rate of convergence in comparison with the direct method with topological enrichment. Significant enhancements are observed for both cases compared with the conventional finite element method with contour integral GSIF estimates.

Fig. 23. Accuracy of GSIF for a coarse mesh (mesh 1.8 × 4.0; 776 nodes, 700 elements). Contour Integral versus domain integral band (left), Collocated GSIF versus enrichment radius (right).

Fig. 24. Accuracy of GSIF for a fine mesh (mesh 0.9 × 2.0; 3138 nodes, 2986 elements). Contour Integral versus domain integral band (left), Collocated GSIF versus enrichment radius (right).

Fig. 25. Comparison of the accuracy of contour integral method with the present direct method.
In Fig. 26, the enrichment radius for geometrical enrichment is $r_e/d = 0.1$ and a $2 \times 2$ quadrature on $10 \times 10$ is performed on enriched and blending elements. The results show that by local enriching the approximation in geometrical enrichment PUFEM, the rate of convergence of the standard Q4 finite element for non-singular problems can be reproduced. The convergence of PUFEM varies averagely between $-0.5$ and $-2.0$ for different quadrature orders and unstructured mesh densities.

The average error of GSIF, determined by the contour integral at $r_{c}/d = 0.05$ (where $r_{c}$ is the contour integral radius and $d$ is the interface width) and the rate of convergence for different Gauss quadrature orders have been compared in Table 3 for the standard finite element simulation.

In order to verify the rate of convergence of contour integral results in enriched PUFEM, different radii of enrichment have been considered. While the GSIF is calculated at the same radius $r_{c}/d = 0.05$, the enrichment is performed over different process zone radiuses of $r_e/d = 0.1, 0.2$. So, the enriched zone covers the contour integral path. The results show that any enrichment of the process zone beyond $r_e/d = 0.1$ has a negligible effect on accuracy and rate of convergence of GSIF generated at radius $r_{c}/d = 0.05$.

The rate of convergence and average error in contour integral GSIF ($r_{c}/d = 0.05, 0.10$), for different quadrature orders are presented in Tables 4 and 5 for $r_e/d = 0.1, 0.2$. The results show a good agreement with [16] with reasonably higher resolution of structured mesh.

The convergence rate versus different Gauss point densities over the enriched nodes is presented in Tables 6 and 7 for the topological and geometrical enrichment PUFEM (direct method), respectively. The convergence curves established for various orders of local integration confirm the convergence rate and mean accuracy of direct GSIF. Comparison of the mean accuracy of GSIF in Tables 3–5, 6 and 7 clearly indicates the superiority of the proposed approach. Mean Error of GSIF in Tables 3–6 is obtained from meshes with number of DOFs between 600 and 6000.

The third numerical example studies the performance of the implemented PUFEM in analyzing a steel punch sliding over a steel substrate and compares it by a non-enriched case with a relatively refined mesh to resemble the exact solution. The example evaluates the variations of GSIF through the loading steps by means of internal stress contours and the rate of convergence of residuals in each increment. The geometric nonlinearity is fulfilled via an updated Lagrangian procedure. During the load increments, top surface of the punch and bottom edge of the substrate is constrained and a compressive load, followed by a series of sway displacement-control loadings are applied to the prescribed nodes of the punch. Outline of the problem is depicted in Fig. 27, with 900 nodes and 803 elements.

Enrichments are applied to nodes within a radius of $r_e = 10$ m. The domain integration approach to derive $K_{GSIF}$ with the contour integral method is performed over a surrounding strip of elements within the enrichment radius. In order to obtain

### Table 3
Comparison of mesh size convergence rates of FEM for GSIF versus quadrature order.

<table>
<thead>
<tr>
<th>Quadrature order, subdivision</th>
<th>$2 \times 5$</th>
<th>$2 \times 10$</th>
<th>$2.15 \times 15$</th>
<th>$2 \times 20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Smoothed convergence rate</td>
<td>$-0.50$</td>
<td>$-0.42$</td>
<td>$-0.32$</td>
<td>$-0.22$</td>
</tr>
<tr>
<td>Mean error of GSIF (%)</td>
<td>74</td>
<td>68</td>
<td>67</td>
<td>67</td>
</tr>
</tbody>
</table>
KGSIF with a maximum 5% error with respect to the direct method, a topological enrichment is adequate. The total number of additional degrees of freedom is 122 (2 per enriched node). Asterisk nodes around the left and right punch corners in Fig. 27 show the enriched nodes. The normal and tangential penalty factors are $5 \times 10^5$. This parameter should be tuned in order to allow reasonably low number of iterations to reach convergence in each increment. A low penalty parameter decreases the accuracy of impenetrability constraint enforcement while high penalty values decrease the rate of convergence through increasing the contact interface residuals and bouncing between the free and slip contact states of slave nodes. The coefficient of friction is considered $\mu_f = 0.3$. Inertia effects are neglected and a quasi static loading sequence is described according to Table 8.

Table 4
Comparison of mesh size convergence rates of PUFEM for GSIF versus quadrature order $r_e/d = 0.05, r_e/d = 0.10$.

<table>
<thead>
<tr>
<th>Quadrature order, subdivision</th>
<th>2, 5 × 5</th>
<th>2, 10 × 10</th>
<th>2, 15 × 15</th>
<th>2, 20 × 20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Smoothed convergence rate</td>
<td>−0.69</td>
<td>−0.93</td>
<td>−0.54</td>
<td>−0.93</td>
</tr>
<tr>
<td>Mean error in GSIF (%)</td>
<td>17</td>
<td>20</td>
<td>18</td>
<td>22</td>
</tr>
</tbody>
</table>

Table 5
Comparison of mesh size convergence rates of PUFEM for GSIF versus quadrature order $r_e/d = 0.10, r_e/d = 0.20$.

<table>
<thead>
<tr>
<th>Quadrature order, subdivision</th>
<th>2, 5 × 5</th>
<th>2, 10 × 10</th>
<th>2, 15 × 15</th>
<th>2, 20 × 20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Smoothed convergence rate</td>
<td>−0.95</td>
<td>−0.59</td>
<td>−0.86</td>
<td>−0.94</td>
</tr>
<tr>
<td>Mean error in GSIF (%)</td>
<td>8</td>
<td>13</td>
<td>11</td>
<td>11</td>
</tr>
</tbody>
</table>

Table 6
Mesh size convergence rates for different quadrature order, the case of the topological enriched PUFEM.

<table>
<thead>
<tr>
<th>Quadrature order, subdivision</th>
<th>2, 5 × 5</th>
<th>2, 10 × 10</th>
<th>2, 15 × 15</th>
<th>2, 20 × 20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convergence rate</td>
<td>−0.10</td>
<td>−0.09</td>
<td>−0.17</td>
<td>−0.10</td>
</tr>
<tr>
<td>Mean error in GSIF (%)</td>
<td>10</td>
<td>8</td>
<td>8</td>
<td>7</td>
</tr>
</tbody>
</table>

Table 7
Mesh size convergence rates for different quadrature order, the case of the geometrical enriched PUFEM ($r_e/d = 0.10$).

<table>
<thead>
<tr>
<th>Quadrature order, subdivision</th>
<th>2, 5 × 5</th>
<th>2, 10 × 10</th>
<th>2, 15 × 15</th>
<th>2, 20 × 20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convergence rate</td>
<td>−1.05</td>
<td>−1.34</td>
<td>−0.81</td>
<td>−1.12</td>
</tr>
<tr>
<td>Mean error in GSIF (%)</td>
<td>6</td>
<td>4</td>
<td>6</td>
<td>4</td>
</tr>
</tbody>
</table>

Fig. 27. Configuration of the Punch sliding problem. fine finite element mesh (left) and geometrically enriched PUFE mesh (right).
The real and imaginary parts of the characteristic equation for leading and trailing corners of steel/steel slave/master cases are described in Fig. 28. The singularity power is $\lambda = 0.5539$ for the leading corner and $\lambda = 0.9799$ for the trailing corner.

The variations of the residuals of the standard FE and geometric enriched PUFEM are illustrated in Fig. 29. In Fig. 29 (right), variations of the residuals of FEM and geometric PUFEM are compared. The results show that adding the enrichment function accelerates the convergence of the Newton–Raphson iterations within each sub-step and decreases the number of iterations to $1/5$ of conventional FEM.

The overall contours of stress and strain fields for a number of successive increments to the end of the procedure are shown in Fig. 30, clearly indicating the singularity nature of all stress components at the contact corners.

Fig. 31 illustrates variations of GSIF for the trailing (left) and leading (right) corners for different loading steps are calculated via the contour integral over 5 m radius band of elements in the FE post-processing stage and collocation unknowns in PUFEM solution.
As explained before, due to extremely local extension of the singular stress field around the vertex point, the interaction contour integral method does not reproduce the exact GSIF. It is observed that the GSIF estimate made by the interaction contour integral method in punch sliding problem is less accurate than that in the corresponding benchmark problem where the asymptotic boundary conditions are enforced on the slave and target boundaries. However, the GSIFs predicted by the direct method on the singular element gives the sole approximate of the real stress intensity factor which is expected to be generated at the singular vertex point. Despite the fact that the reliability of the GSIF results depends on the method of constraint enforcement (penalty, Lagrange multipliers, etc.), the predictions by the proposed method have the advantage of being independent of the radius of contour integral and the enrichment band radius and removes the cost of post-processing calculations around the singular point to estimate the failure design parameters (i.e. GSIF), because an unconditionally stable result is derived directly over the collocated enriched degrees of freedom.

6. Conclusion

The consistent asymptotic fields for a sliding punch corner have been solved numerically and applied as enrichment functions within a PUFEM framework. It is indicated that implementation of the enrichments in the compatible form, which sat-
isfies the local boundary conditions, increases the accuracy of GSIF derived from the contour integral method, with minimal additional degrees of freedom. Furthermore, the proposed definition of enrichment function eliminates the need for facilitating contour integral method in the post-processing phase, because GSIF can be computed accurately, as collocated additional degrees of freedom located on the vertex point and its neighboring nodes and estimates the exact value with a negligible error. The effects of topological and geometric enrichments on the rate of convergence of the energy error norm and GSIF versus different local element size have been considered. Moreover, it is shown that the rate of convergence of the Newton Raphson iterations is improved in PUFEM approach. Variation of GSIF in a real sliding punch problem for different stages of loading is derived and superiority of the consistent enrichment function over previously available incompatible enrichment strategy is demonstrated. For other cases of contact problems including sharp corners with different boundary conditions such as slip-free indentation of cutting tools, symmetric sliding cone penetration, where neither a closed form solution nor a definite reduced form of the characteristic equation is addressed in literature, implementation of the present PUFEM remains applicable and straightforward.

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References
