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Analytical derivation of tortuosity and permeability of monosized spheres: A volume averaging approach

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Macroscopic properties of granular materials are important in modeling a variety of flow and transport phenomena in many fields of science. Determination of these parameters has always been an issue among both researchers and engineers, mainly in view of tortuosity and permeability. This paper presents analytical functions for the tortuosity and permeability of monosized sphere arrays based on a volume averaging approach and eliminates some ambiguities by modification of the original representative elementary volume model. Veracity of the proposed formulations has been illustrated through comparisons with the latest available results on the subject. Good agreement is found.

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I. INTRODUCTION

Since the pioneering work of Kozeny [1], a vast number of researchers and engineers have dealt with establishing relations for permeability K of porous media. Carman [2–4] modified the original equation of Kozeny to the following form:

$$K = \frac{n^3}{k(1-n)^2 S^2}, \tag{1}$$

where n is the porosity, S is the specific surface of the solid volume, and $k = C_f \tau^2$ is the Kozeny-Carman (KC) constant, in which τ is the tortuosity, described as the ratio of the actual tortuous length of flow path to the shortest straight distance along the macroscopic pressure gradient and C_f is a constant depending on capillary pore shape. Assuming spherical particles, substituting for S in Eq. (1) leads to

$$K = \frac{n^3}{36 C_f \tau^2 (1-n)^2} d_p^2, \tag{2}$$

where d_p is the particle diameter. A value of $C_f = 2.5$ is given for beds of spherical particles [5]. Carman proposed a constant tortuosity of $\tau = \sqrt{2}$ in Eq. (2) based on his experimental measurements of permeability [2]. As a result, the most widely used form of the KC correlation for a monodisperse sphere packing is obtained [5–8]:

$$K = \frac{d_p^2}{180} \frac{n^3}{(1-n)^2}, \tag{3}$$

which is considered as a simple, yet practical, correlation for expressing the permeability of granular media in terms of particle size and porosity.

Despite its extensive application, the KC constant has been realized not to be a constant for different porous media; for example, Mathavan and Viraraghavan [9] proposed a value of 3.4 for peat beds, contrary to 4.8 ± 0.3 by Carman [4] for uniform spheres. Rahli *et al.* [10] determined different KC constants for the randomly packed monodispersed fibers at different porosities and aspect ratios. A brief review of some permeability equations with variable KC constants is provided for some porous media, e.g., textile assemblages, fiber mats, particle arrays, rocks, and unconsolidated porous media [11]. The review showed the vast scatter of permeability equations in the literature pertaining to different types of porous media. This clarifies the effect of structure of the porous media on permeability, which will not be discussed in this paper.

Not only does the KC constant depend on the porous structure, but also it varies with porosity, tortuosity and pore area for the same medium. The variation is extensive in a way that many existing permeability equations are only valid at certain porosity ranges [11]. Rumpf and Gupte [12], following experiments on beds of spherical particles, presented a permeability relation for porosities between 0.35 and 0.70. Howells [13] and Hinch [14] expanded the Brinkman [15] correlation and suggested another permeability equation valid for porosities above 0.75 for equal-sized sphere suspensions. Sriboonlue and Davies [16] proposed a KC constant for coarse gravels with a porosity range of 0.336 and 0.400 based on experiments. Bryant and Blunt [17] simulated dense packing of monosized spheres using a pore-network model for $n = 0.362$. Maier *et al.* [18] and Guodong *et al.* [19] utilized the lattice-Boltzmann method to obtain permeability of identical spheres at $n = 0.37$. All these results were comparable to the KC equation. Garcia *et al.* [20] achieved similar, though slightly higher, permeabilities for the mentioned porosities. Experimental evidences illustrated that the KC equation provides reasonable permeability estimates for random packing of spheres, periodic arrays of spheres, and fractal porous media [21]. On a comprehensive study on the subject using direct three-dimensional (3D) computational fluid dynamics (CFD) model of monosized spheres, Zaman and Jalali [22] quoted that the KC equation has an extended range of agreement with simulations up to a porosity of 0.9 when 1000 or higher numbers of particles were utilized. The higher the number of spheres, the more homogenous the medium became from a microstructural point of view (especially at higher porosities).

Despite the universal consensus on the validity of the KC equation for monosized spheres, as indicated above, there

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TABLE I. Some reference tortuosity functions for particulate media.

Reference	Tortuosity correlation	Porosity range	Applicability	Derivation method
Comiti and Renaud [23]	$1 - p\ \ln(n)$	–	Bed of particles	Experimental (conductivity measurements)
Koponen *et al.* [24]	$1 + p(1 - n)$	$0.5 < n < 1$	–	Numerical (2D lattice-Boltzmann method)
Weissberg [25], Mauret and Renaud [26], and Barrande *et al.* [27]	$1 - 0.49\ \ln(n)$	$0.36 < n < 1$	Bed of spheres	Experimental (conductivity measurements)
Iversen and Jørgensen [28]	$\sqrt{1 + 2(1 - n)}$	$0.4 < n < 0.9$	Sandy marine sediments	Experimental (diffusion measurements)
Bear [6], Dullien [7], Mota *et al.* [29], and Dias *et al.* [30,31]	$\frac{1}{n^p}$	–	Granular beds	Experimental (conductivity measurements)

is a large scatter in existing tortuosity correlations, especially in their form and porosity range of applicability. Some of the most cited tortuosity functions available in the literature for particulate media are as follows:

Comiti and Renaud [23]:

$$\tau = 1 - p\ \ln(n); \tag{4}$$

Koponen *et al.* [24]:

$$\tau = 1 + p(1 - n); \tag{5}$$

Weissberg [25], Mauret and Renaud [26], and Barrande *et al.* [27]:

$$\tau = 1 - 0.49\ \ln(n); \tag{6}$$

Iversen and Jørgensen [28]:

$$\tau = \sqrt{1 + 2(1 - n)}; \tag{7}$$

Bear [6], Dullien [7], Mota *et al.* [29], and Dias *et al.* [30,31]:

$$\tau = \frac{1}{n^p}. \tag{8}$$

These functions, together with their porosity range, applicability, and derivation method, are summarized in Table I. Most of the functions are derived from physical or numerical experiments and do not possess a strong analytical basis. Table I shows that for identical methods of derivation, the correlations have different forms. For example, although Eqs. (6) and (8) are both validated in conductivity experiments and are highly accredited by several researchers, they have different mathematical forms. The same inconsistency is observed for correlations derived from numerical simulations. Ghassemi and Pak [32] have suggested a correlation of the form given by Eq. (5) from two-dimensional (2D) simulations using the lattice-Boltzmann method, which agrees [24], while similar numerical simulations by Matyka *et al.* [33] have shown a correlation of the form $1 + \ln(n)$, which is different from Eq. (5). However, the latter may be attributed to overlapping of the rectangles, which was allowed in Matyka *et al.* [33] simulations.

As a result, 2D numerical models cannot account for tortuosity variations. Because of the 3D nature of tortuosity, it may not be limited to in-plane flow paths. The 3D numerical flow simulations are usually computationally cumbersome.

It is apparent from the above that tortuosity has many definitions that were derived from different methods and it is not clear if they are equivalent. For instance, computer simulations (e.g., Ref. [24]) calculate tortuosity from flow streamlines, and it is not obvious that a quantity thus defined should have much in common with the various tortuosities calculated from experiments (e.g., diffusion or electric tortuosity), as the only link between them is a very simple theory that views a porous medium as composed of separate tubes of constant cross section. A thorough discussion in this regard can be found in Ref. [34].

The role of analytical methods is, therefore, to shed light on the subject and to build up a mathematical framework for the problem at hand. In spite of the possible simplicity of such solutions, they supply the required framework for the researchers to adapt the inherent coefficients to suit experiments.

To the best of the authors' knowledge, a reliable analytical framework for tortuosity has not been suggested so far. Weissberg [25] presented a diffusivity model for freely overlapping spheres. Their proposed function was similar to Eq. (6), which was later validated empirically [26,27], even though the assumption of overlapping spheres is unrealistic. Du Plessis and Masliyah [35] provided the following analytical function for isotropic granular media based on the concept of a representative unit cell (RUC) [36]:

$$\tau = \frac{n}{1 - (1 - n)^{2/3}}. \tag{9}$$

RUC is a concept in which pores and the representative elementary volume (REV) are represented by a cubic volume of minimum dimensions.

Figure 1 compares Eq. (9) to the reference tortuosity functions of Table I. Deviation, both qualitatively and quantitatively, is obvious. However, it satisfies the limit case of $\tau \to 1$ when $n \to 1$.

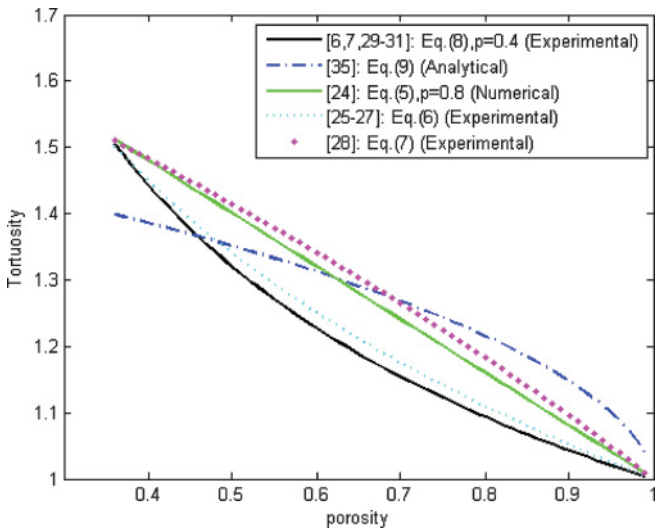

FIG. 1. (Color online) Comparison of some existing analytical tortuosity functions to those from experiments.

On a recent attempt, Lanfrey *et al.* [37] proposed the following analytical function for sphere beds:

$$\tau = 1.23\frac{(1-n)^{4/3}}{n^2}. \tag{10}$$

This tortuosity function not only generates quantitatively invalid values, but also does not fulfill the limit case of unit tortuosity when porosity approaches one (it tends to zero instead). In order to avoid scale disturbances to diagrams of Fig. 1 (owing to its out-of-range values), it is not plotted in Fig. 1. As pointed out by the authors themselves, this error is on account of assuming a constant value of 1.23 in Eq. (10) , which restricts the possible macroscopic packing data.

The present paper aims at providing analytical tortuosity and permeability functions for sphere arrays based on the concept of REV, and to compare them with the existing correlations.

II. REPRESENTATIVE ELEMENTARY VOLUME (REV) MODEL

Since the early Biot's coupled formulation of Darcy flow in porous media [38], many improvements have been proposed and employed successfully over the years in geomechanical software packages. The formulation was based on only two macroscopic parameters: porosity and permeability [39–41]. An extensive review of the methods can be found in Ref. [42].

Subsequent models used an approach of local volume averaging for deriving balance equations of the porous medium. Among them is the model presented by Bear and Bachmat [43–45], which provides a mathematical continuum framework for addressing macroscopic parameters of the porous medium. The model handles a macroscopic configuration of granules through tensorial parameters. Some basic definitions of the model are reproduced in Fig. 2, in which $\beta = s$ denotes solid particles and $\alpha = f$ specifies fluid saturating the voids. In the following, a two-phase porous medium is assumed, as depicted in Fig. 2, where s and f subscripts represent solid and fluid phases, respectively. In

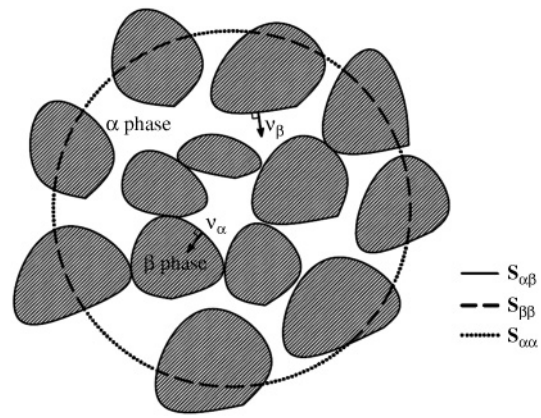

FIG. 2. REV defined in Ref. [43]; $\alpha \equiv f$; $\beta \equiv s$; $S_0 = S_{ff} + S_{ss}$; $S_{0f} = S_{fs} + S_{ff}$.

the figure, S_0 indicates the whole area encompassing the REV, S_{ss} denotes part of the area around the REV that passes through the granules, S_{ff} represents part of the area around the REV that does not cross the granules, and S_{fs} describes the area in which the granules and the fluid interact with each other. S_{0f} stands for the area enclosing the fluid volume, which is equal to the sum of S_{ff} and S_{fs}. Also,

$$\overline{(W)}^{\chi} = \frac{1}{U_{0\chi}}\int_{U_{0\chi}} W\,dU \tag{11}$$

is the volumetric intrinsic phase average of a quantity W over phase χ (s or f), where $U_{0\chi}$ is the volume occupied by phase χ. The $\overline{(\,)}^{\chi}$ denotes the volumetric intrinsic phase average over phase χ. Also,

$$\overline{(W)} = \frac{1}{U_0}\int_{U_{0\chi}} W\,dU \tag{12}$$

is the volumetric phase average of a quantity W over phase χ (s or f), where U_0 is the total volume occupied by the solid and fluid phases, i.e., $U_0 = U_{0s} + U_{0f}$. The $\overline{(\,)}$ denotes the volumetric phase average over phase χ. Terms without the overbar ($\bar{\ }$) denote microscopic (nonaveraged) quantities.

Bear and Bachmat [45] presented macroscopic balance equations of a porous medium. In order to express permeability of the fluid, they used the concept of Darcy flow, in which drag forces in the interface of fluid and solid were assumed to be dominant over inertia and viscous forces of the fluid in the macroscopic fluid momentum balance equation. This can be written for the case of an isotropic porous medium as [45]

$$\overline{\mu_f}^{f}\alpha_{ij}C_f n\frac{\left(\overline{V_{f_j}}^{f} - \overline{V_{s_j}}^{s}\right)}{\Delta_f^2} = -nT^*_{fji}\left(\frac{\partial\overline{p_f}^{f}}{\partial x_j} + \overline{\rho_f}^{f} g\frac{\partial z}{\partial x_j}\right), \tag{13}$$

where i and j are free and dummy indexes, respectively; μ_f is the dynamic viscosity of the fluid; C_f is a shape factor related to pore size [same as Eq. (2)]; V_i, ρ, and p denote velocity, density, and pressure, respectively; g is the gravity acceleration; and z is the vertical coordinate (assumed positive upward). All four macroscopic parameters of the model, which appear in Eq. (13), are defined as follows [45]:

(1) Porosity n denotes the fraction of fluid in a REV, i.e.,

$$n = \frac{U_{0f}}{U_0}. \tag{14}$$

(2) Hydraulic radius Δ_f is defined as the ratio of fluid volume to the fluid-solid interface area:

$$\Delta_f = \frac{U_{0f}}{S_{fs}}. \tag{15}$$

where S_{fs} is the interface area of the fluid and solid phases (Fig. 2).

(3) The tortuosity tensor is defined as

$$T^*_{fij} = \frac{1}{U_{0f}} \int_{S_{ff}} \mathring{x}_i \nu_{fj}\, dS, \tag{16}$$

where S_{ff} is the area of part of circumference of the REV intercepted by the fluid, ν_{fi} is the unit normal to the surface (see Fig. 2), and $\mathring{x}_i = x_i - x_{0i}$, in which x_0 is the coordinate of center of the REV.

It was shown in Ref. [45] that the tortuosity tensor for an isotropic medium can be written in the following form:

$$T^*_{fij} = T^*_f \delta_{ij} = \frac{\theta^S}{n}\delta_{ij}, \tag{17}$$

where δ_{ij} is the Kronecker delta function, and $\theta^S = \frac{S_{ff}}{S_0}$.

(4) α_{ij} is another second-rank tensor that depends on the granules configuration in the Bear and Bachmat model [45]:

$$\alpha_{ij} = \delta_{ij} - \widetilde{\nu_i \nu_j}^{fs} = \delta_{ij} - \frac{1}{S_{fs}} \int_{S_{fs}} \nu_{fi}\nu_{fj}\, dS. \tag{18}$$

The $(\tilde{\ })^{fs}$ defined above is the average over S_{fs}. For an isotropic medium,

$$\alpha_{ij} = a\delta_{ij}. \tag{19}$$

Combining with (13), and after some algebraic manipulations,

$$\overline{V_{f_i}} - \overline{V_{s_i}} = -\frac{n\Delta_f^2}{C_f} T^*_f a^{-1} \frac{\gamma_f}{\overline{\mu_f}^f} \left(\frac{\partial\left(\frac{\overline{p_f}^f}{\gamma_f}\right)}{\partial x_i} + \frac{\partial z}{\partial x_i} \right). \tag{20}$$

Equation (20) is the Darcy equation, from which fluid conductivity is found:

$$k_f = \frac{n\Delta_f^2}{C_f} T^*_f a^{-1} \frac{\gamma_f}{\overline{\mu_f}^f}. \tag{21}$$

By analogy with $k_f = K\frac{\gamma_w}{\mu_f}$, the (intrinsic) permeability becomes

$$K = \frac{n\Delta_f^2}{C_f} T^*_f a^{-1}, \tag{22}$$

which is a macroscopic parameter that, in addition to being independent of fluid properties, encompasses all four principle macroscopic parameters of the model.

Bear and Bachmat [45] then presented that the following simplifications are relevant in the case of isotropy,

$$\begin{aligned} \alpha_{ii} &= 3a = 3 - \nu_i\nu_i = 3 - \underbrace{(\nu_1\nu_1 + \nu_2\nu_2 + \nu_3\nu_3)}_{1} \\ &= 2 \Rightarrow a = \frac{2}{3} \end{aligned} \tag{23}$$

and

$$K = \frac{3}{2}\frac{n\Delta_f^2}{C_f} T^*_f. \tag{24}$$

The present study shows that the assumption of independent constant a parameter in Eq. (23) is not necessary for isotropic media. In other words, it can be related to the tortuosity, as described subsequently.

We begin with

$$\int_{S_{fs}} \mathring{x}_i \nu_{fj}\, dS + \int_{S_{ff}} \mathring{x}_i \nu_{fj}\, dS = \int_{S_{0f}} \mathring{x}_i \nu_{fj}\, dS. \tag{25}$$

Because S_{0f} is a closed surface with no internal singularity, the Gauss theorem can be applied to the right-hand side of Eq. (25), which yields

$$\int_{S_{fs}} \mathring{x}_i \nu_{fj}\, dS + \int_{S_{ff}} \mathring{x}_i \nu_{fj}\, dS = \int_{U_{0f}} \delta_{ij}\, dU_f = U_{0f}\delta_{ij}. \tag{26}$$

Dividing Eq. (26) by U_{0f} and using Eq. (16), it transforms to

$$\frac{1}{U_{0f}} \int_{S_{fs}} \mathring{x}_i \nu_{fj}\, dS = \delta_{ij} - T^*_{f_{ij}}. \tag{27}$$

The integral on the left-hand side of the Eq. (27) expresses the total static moment of the oriented elementary surfaces comprising the S_{fs} surface with respect to planes passing through the centroid of the REV. To calculate this integral, a hypothetical spherical REV of radius R is assumed. In this way, using $\Delta_f = R/3$, the integral becomes

$$\int_{S_{fs}} \mathring{x}_i \nu_{fj}\, dS = \int_{S_{fs}} R\nu_{fi}\nu_{fj}\, dS = 3\Delta_f \int_{S_{fs}} \nu_{fi}\nu_{fj}\, dS. \tag{28}$$

Substituting in (27) leads to

$$\int_{S_{fs}} \nu_{fi}\nu_{fj}\, dS = \frac{U_{0f}}{3\Delta_f}(\delta_{ij} - T^*_{f_{ij}}). \tag{29}$$

Also from Eq. (18),

$$\int_{S_{fs}} \nu_{fi}\nu_{fj}\, dS = S_{fs}(\delta_{ij} - \alpha_{ij}). \tag{30}$$

The left-hand sides of the Eqs. (29) and (30) are equal, thus

$$\delta_{ij} - \alpha_{ij} = \frac{U_{0f}}{3S_{fs}\Delta_f}(\delta_{ij} - T^*_{f_{ij}}). \tag{31}$$

By the definition of hydraulic radius in Eq. (15), Eq. (31) can be simplified:

$$\alpha_{ij} = \frac{2}{3}\delta_{ij} + \frac{T^*_{f_{ij}}}{3}. \tag{32}$$

For an isotropic medium, Eqs. (19) and (17) remain valid. Therefore, Eq. (32) becomes

$$\alpha_{ij} = \left(\frac{2}{3} + \frac{T^*_f}{3}\right)\delta_{ij}, \tag{33}$$

and using Eq. (19), a new expression for an isotropic medium

is obtained:

$$a = \frac{2}{3} + \frac{T_f^*}{3}. \tag{34}$$

Thus, Eq. (34) indicates that α_{ij} is not constant and depends on tortuosity. Therefore, variation of tortuosity has a direct influence on α_{ij}. In other words, the assumption of Eq. (23) leads to $T_f^* = 0$, according to Eq. (34), which is not accurate.

In order to investigate the effect of the newly developed equation on permeability, Eq. (34) is substituted in Eq. (22) to derive an improved permeability equation:

$$K = \frac{3n\Delta_f^2}{C_f}\frac{T_f^*}{2+T_f^*}. \tag{35}$$

Comparing Eq. (35) to that of Ref. [45] in Eq. (24) reveals a double effect of tortuosity in the permeability equation. In contrast to the previous assumption that permeability is a linear function of tortuosity, it is now found to be a hyperbolic type.

A simple REV model for granular medium is now assumed in order to provide a quantitative measure of the proposed equation.

III. SIMPLE REV MODEL

A simple REV model of regular cubic array of monosized spheres of diameter d_p is illustrated in Fig. 3. The sphere configuration in the cubic array of Fig. 3 is not fixed, and porosity of the model is assumed to vary according to the center-to-center distance of the spheres, denoted by S, in an isotropic manner. In fact, the minimum porosity of the array is obtained for $S = d_p$, that is, $n = 1 - \frac{\pi}{6} \simeq 0.48$, in which the spheres are in contact. As the porosity increases, the spheres lose contact and S becomes large. Therefore, the simple cubic REV of Fig. 3 can be used for monosized spheres with porosities between 0.48 and 1.

Dimensions of the medium are considered infinite with respect to sphere diameters, so that finite-size effects are negligible. From the definitions in the previous section, the following can be written for the REV model:

$$U_0 = S^3, \tag{36}$$

$$U_{0f} = S^3 - 8\left(\frac{\pi d_p^3}{6}\frac{1}{8}\right) = S^3 - \pi d_p^3/6, \tag{37}$$

$$S_0 = 8S^2, \tag{38}$$

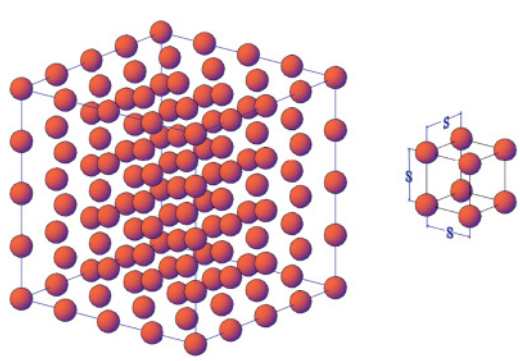

FIG. 3. (Color online) Cubic array of monosized spheres.

$$S_{fs} = 8\left(\pi d_p^2/8\right) = \pi d_p^2, \tag{39}$$

$$S_{ff} = 8\left(S^2 - 4\pi d_p^2/16\right) = 8S^2 - 2\pi d_p^2. \tag{40}$$

Subsequently, the macroscopic parameters of the REV model are calculated based on Eqs. (36)–(40).

Following the definition of porosity by Eq. (14),

$$n = 1 - \frac{\pi}{6}\left(\frac{d_p}{S}\right)^3, \tag{41}$$

from which

$$S = \frac{d_p}{\sqrt[3]{\frac{6}{\pi}(1-n)}}. \tag{42}$$

The hydraulic radius is derived from Eqs. (15) and (42) and some manipulations:

$$\Delta_f = \frac{d_p}{6}\left(\frac{n}{1-n}\right). \tag{43}$$

T_f^* is calculated by Eqs. (17) and (42) as

$$T_f^* = \frac{1 - B(1-n)^{2/3}}{n}, \tag{44}$$

where for the regular array of spheres,

$$B = \frac{\pi}{4}\left(\frac{6}{\pi}\right)^{2/3} = 1.209. \tag{45}$$

Replacing Eqs. (43) and (44) in Eq. (35) yields the conventional form of the permeability equation:

$$K = \frac{[1 - 1.209(1-n)^{2/3}]}{12C_f[1 - 1.209(1-n)^{2/3} + 2n]}\frac{n^3}{(1-n)^2}d_p^2. \tag{46}$$

By analogy with Eq. (2), the tortuosity function is expressed by

$$\tau = \sqrt{\frac{2n}{3[1 - 1.209(1-n)^{2/3}]} + \frac{1}{3}}. \tag{47}$$

Adopting $C_f = 2.5$, as recommended by Kaviany [5] for beds of spherical particles, the final form of the newly developed permeability equation is written as

$$K = \frac{[1 - 1.209(1-n)^{2/3}]}{30[1 - 1.209(1-n)^{2/3} + 2n]}\frac{n^3}{(1-n)^2}d_p^2. \tag{48}$$

For comparison purposes, the permeability equation (24) by Bear and Bachmat [45] after substituting in Eqs. (43) and (44) becomes

$$K = \frac{1 - 1.209(1-n)^{2/3}}{24C_f n}\frac{n^3}{(1-n)^2}d_p^2 = \frac{1 - 1.209(1-n)^{2/3}}{60n}\frac{n^3}{(1-n)^2}d_p^2, \tag{49}$$

from which a 'hypothetical' tortuosity function by Bear and Bachmat [45] is calculated:

$$\tau = \sqrt{\frac{2n}{3[1 - 1.209(1-n)^{2/3}]}}. \tag{50}$$

It should be emphasized that Eqs. (49) and (50) really have not been suggested by Bear and Bachmat [45]; they only have been computed and mentioned here to provide quantitative measure over the improvements made by the present study. The term "hypothetical" denotes this. This will be further described in the next section.

IV. VERIFICATION

To illustrate the performance of the model and the proposed equation, the present tortuosity function of Eq. (47) is compared with some accredited functions in Fig. 4. The hypothetical equation of Bear and Bachmat [45] in Eq. (50) also has been incorporated in the comparison to show the effect of the improvements made by the present study. Figure 4 clearly shows very good agreement of the present tortuosity function with the literature data. The graph is nearly linear in the range of porosities between 0.48 and 0.64, which matches Refs. [6,7,29–31]. However, as the porosity increases, it differs slightly from Eq. (8), and the slope of the diagram decreases until porosities of ∼0.95, where it approaches Eqs. (5) and (7) by Refs. [24] and [28].

The apparent discrepancy in Fig. 4 for porosities lower than 0.48 is owing to limitation of the cubic sphere model, in which the minimum porosity applicable is $1 - \pi/6 = 0.4764$ for $S = d_p$.

The hypothetical equation of Bear and Bachmat [45] is also shown in Fig. 4. The difference between Eqs. (47) and (50) becomes more protrudent as porosity increases. Also, Eq. (50) does not meet the requirement of $\tau = 1$ at $n = 1$. This is the result of assuming a constant value for the α_{ij} tensor, as described in Sec. II. It is seen that the modification of this tensor improves the tortuosity prediction.

A similar comparison for the proposed permeability equation (48) is plotted in Fig. 5. Again, good coherence with the KC correlation is observed. Although the present model predicts a more consistent permeability than Bear and Bachmat [45], the difference remains limited. Therefore, it is inferred that considering a variation of the α_{ij} tensor does not substantially contribute to enhance the permeability prediction.

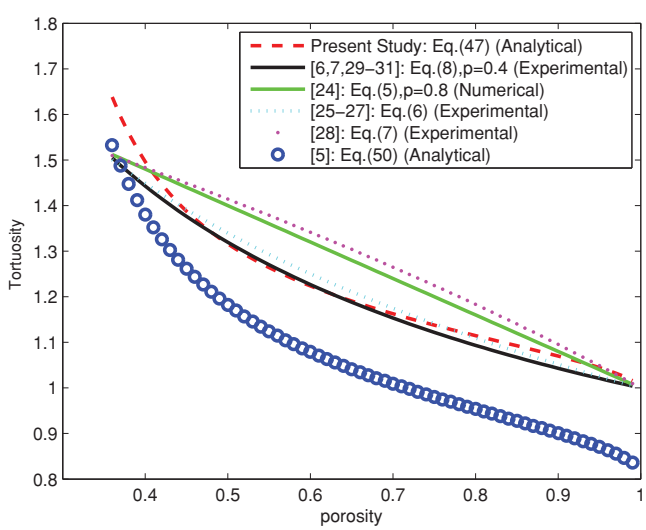

FIG. 4. (Color online) Graphical comparison of the tortuosity functions.

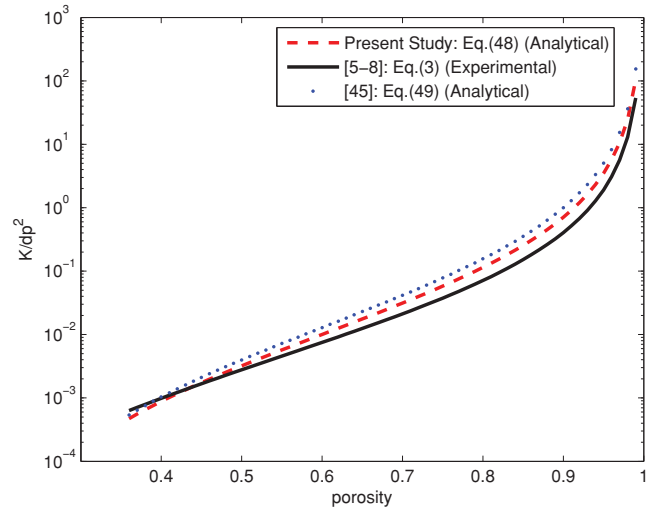

FIG. 5. (Color online) Graphical comparison of the permeability functions.

V. DISCUSSION

The REV model introduced in Sec. III to demonstrate the influence of the theoretical developments on tortuosity and permeability is an ideal packing in which the spheres do not lose their cubic configuration even when porosity increases. Even though this assumption is not generally true, it was utilized to provide a quantitative measure for comparison purposes. However, a question may arise here concerning the effect of the packing configuration in the tortuosity and permeability predictions of the present theory.

In this regard, the tetrahedral REV illustrated in Fig. 6 is considered. Again, dimensions of the medium are considered infinite with respect to sphere diameters, so that finite-size effects are negligible. The REV allows for the minimum porosity obtainable for a packing consisting of monosized spheres for $S = d_p$. Porosity of the REV increases as the spheres become distant and S grows.

Applying a similar procedure to the one adopted in Sec. III for the REV of Fig. 6 yields

$$B = \left(\frac{3}{16}\right)^{1/6} \pi^{1/3} \simeq 1.108, \tag{51}$$

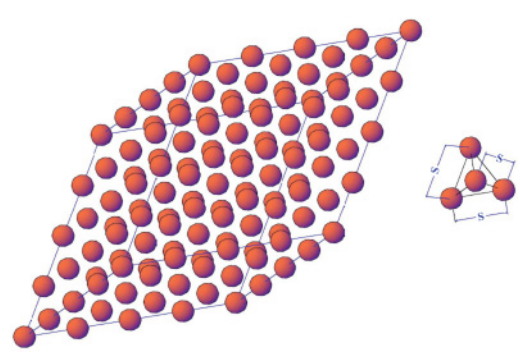

FIG. 6. (Color online) Tetrahedral array of monosized spheres.

TABLE II. Summary of B values for the cubic and tetrahedral packings of monosized spheres.

Packing	B
Cubic	1.209
Tetrahedral	1.108

$$\tau = \sqrt{\frac{2n}{3[1-1.108(1-n)^{2/3}]} + \frac{1}{3}}, \tag{52}$$

$$K = \frac{[1-1.108(1-n)^{2/3}]}{30[1-1.108(1-n)^{2/3}+2n]}\frac{n^3}{(1-n)^2}d_p^2. \tag{53}$$

The calculations performed for the two different REVs based on the mathematical framework developed in this study reveal that tortuosity and permeability functions for a granular medium composed of identical spheres may be presented in the form

$$\tau = \sqrt{\frac{2n}{3[1-B(1-n)^{2/3}]} + \frac{1}{3}}, \tag{54}$$

$$K = \frac{[1-B(1-n)^{2/3}]}{30[1-B(1-n)^{2/3}+2n]}\frac{n^3}{(1-n)^2}d_p^2, \tag{55}$$

in which B values for the two ideal cubic and tetrahedral packings of spheres are summarized in Table II.

It may be inferred that the geometric arrangement of the spheres shows its influence in the equations through the variable B. Despite being assumed to be constant in the two REVs considered, this parameter is not actually a constant; however, the analytical framework presented in this paper allows for a different variable to be employed in experimental correlations for tortuosity and permeability to account for macroscopic packing of the spheres.

VI. CONCLUSION

After modification of a macroscopic tensor in the Bear and Bachmat [45] model, analytical expressions for tortuosity and permeability have been proposed for a simple REV of cubic array of spheres. The developed analytical expressions were verified by comparing them with well-recognized functions, for which very good agreement was evident.

The effect of the adjustment made to α_{ij} was also investigated. It is found that taking into account the variations of the tensor has considerable effect on the tortuosity; however, the effect on the permeability was not remarkable, though slightly advantageous.

In conclusion, a notable achievement of the present study is the development of tortuosity and permeability functions that are not only completely analytical but also cohere well with the empirical data. Besides, it provides an analytical framework for experimental correlations to describe tortuosity and permeability of the granular media considering the packing structure of the particles. To the best of authors' knowledge, no such successful attempt has been made so far.

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