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Dynamic analysis of fixed cracks in composites by the extended finite element method

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This paper is dedicated to simulation of dynamic analysis of fixed cracks in orthotropic media using an extended finite element method. This work is in fact an extension to dynamic problems of the recently developed orthotropic extended finite element method for fracture analysis of composites. In this method, the Heaviside and near-tip enrichment functions are used in the framework of the partition of unity for modeling crack discontinuity and crack-tip singularities within the classical finite element method. In this procedure, elements that include a crack are not required to conform to crack edges. Therefore, mesh generation can be performed without any need to comply to crack edges and the method is capable of modeling the crack propagation without any remeshing. To determine the fracture properties, mixed-mode dynamic stress intensity factors (DSIFs) are evaluated by means of domain separation integral (J-integral) method. Results of the proposed method are compared with other available analytical and computational results.

1. Introduction

Importance and wide application of composites in many industries such as aerospace and automotive, have led to the growing need for better understanding of the behavior of composite materials. Safe and effective uses of these materials necessitate a great need for analyzing their behaviors in critical conditions.

Accordingly, many investigations have been performed in recent decades in order to simulate and analyze the complex behavior of these materials. Muskhelishvili [1], Savin [2], Lekhnitskii [3], Sih et al. [4], Tuholme [5] and Ting [6] tried to solve a number of elasticity problems that were related to discontinuity in anisotropic elastic bodies with mathematical/analytical approaches such as complex functions. Viola et al. [7], Broberg [8], Lim et al. [9] and Nobile and Carloni [10] have extensively contributed to the analytical solutions of crack propagation and dynamic crack analysis by implementing boundary value problems to solve several basic problems in anisotropic media.

Based on the fact that analytical methods are not versatile enough to solve arbitrary problems, numerical methods such as the boundary element method, the finite element method, and mesh-less methods have been developed and implemented in various engineering problems. Aliabadi et al. [11] and García-Sánchez et al. [12] developed boundary element solutions for crack propagation and dynamic analysis of crack in orthotropic media. Meanwhile, Belytschko et al. [13,14] have performed many investigations to solve and improve dynamic fracture problems with the element-free Galerkin method. However, boundary element methods cannot be readily extended to non-linear systems, while the majority of mesh-less methods are numerically expensive or may require difficult stabilization schemes. In contrast, the finite element method is well developed into the non-linear problems and can be easily adapted to many types of boundary conditions and geometries.

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Among various finite element formulations, singular finite elements have been widely used for fracture analysis of structures [15,16]. For crack simulations, however, finite element meshes must adapt to crack edges and remeshing should be utilized when a crack propagates.

The extended finite element method (XFEM) resolves this deficiency by enriching the finite element model in the vicinity of the crack. This method allows the domain to be simulated without explicitly meshing the crack. It was originally proposed by Belytschko and Black [17] and then improved by Moes et al. [18]. Subsequently, more investigations have been carried out; including Sukumar et al. [19] and Areias and Belytschko [20] extending the method to three dimensional domain, Sukumar and Prévost [21] describing computer implementation of the method, Dolbow et al. [22] using frictional contact, Mergheim et al. [23] simulating cohesive cracks and Belytschko et al. [24,25], Grégoire et al. [26] and Prabel et al. [27] modeling dynamic crack propagation for isotropic materials.

Recently, Asadpoure et al. [28,29] proposed a number of formulations for crack simulations in orthotropic materials by XFEM. First, they improved the crack-tip enrichment functions for each class of composites [28,29], and then developed a unified set of crack-tip enrichment functions for all composites [30].

In this study, the existing static orthotropic XFEM solution is further extended for analyzing the dynamic stress intensity factors (DSIFS) are obtained and compared with other available data in a set of numerical problems.

2. Fracture mechanics for 2D orthotropic materials

The Hooke’s law between linear strain tensor \( e_{ij} \) and stress tensor \( \sigma_{kl} \) in an arbitrary linear elastic homogeneous material can be written as

\[
e_{ij} = s_{ijkl} \sigma_{kl} \quad (i,j,k,l = 1, 2, 3)
\]

where \( s_{ijkl} \) is the fourth-order compliance tensor with 21 independent coefficients. The stress–strain relation can then be rewritten in the following contracted form proposed by Lekhnitskii [3]:

\[
e_x = a_{x\beta} \sigma_\beta \quad (\alpha, \beta = 1 - 6)
\]

with

\[
e_1 = e_{11}, \quad e_2 = e_{22}, \quad e_3 = e_{33}, \quad e_4 = 2e_{23}, \quad e_5 = 2e_{31}, \quad e_6 = 2e_{12}
\]

\[
\sigma_1 = \sigma_{11}, \quad \sigma_2 = \sigma_{22}, \quad \sigma_3 = \sigma_{33}, \quad \sigma_4 = \sigma_{23}, \quad \sigma_5 = \sigma_{31}, \quad \sigma_6 = \sigma_{12}
\]

and coefficients \( a_{x\beta} \) are defined in terms of compliance coefficients \( s_{ijkl} \)

\[
\begin{cases}
    a_{x\beta} = s_{ijkl} & \text{both } \alpha, \beta \leq 3 \\
    a_{x\beta} = 2s_{ijkl} & \alpha \leq 3 \text{ and } \beta > 3 \\
    a_{x\beta} = 2s_{ijkl} & \alpha > 3 \text{ and } \beta \leq 3 \\
    a_{x\beta} = 4s_{ijkl} & \text{both } \alpha, \beta > 3
\end{cases}
\]

\[
\alpha = \begin{cases}
    i, & \text{if } i = j \\
    9 - i - j, & \text{if } i \neq j
\end{cases}, \quad \beta = \begin{cases}
    k, & \text{if } k = l \\
    9 - k - l, & \text{if } k \neq l
\end{cases}
\]

In a plane stress state, Eq. (2) depends only to the following independent parameters

\[
a_{ij} \quad (i,j = 1, 2, 6)
\]

and for a plane strain state, \( a_{ij} \) are replaced by the following parameters,

\[
\left[ a_{ij} \right] = \left[ a_{ij} - \frac{a_{13} - a_{23}}{a_{13}} \right] \quad (i,j = 1, 2, 6)
\]

For a crack in an anisotropic body with general boundary conditions and subjected to arbitrary forces, the characteristic Eq. (9) was introduced by Lekhnitskii [3] using equilibrium and compatibility conditions

\[
a_{11} \mu^2 - 2a_{16}\mu^3 + (2a_{12} + a_{66})\mu^2 - 2a_{26}\mu + a_{22} = 0
\]

Lekhnitskii [3] illustrated that the roots of Eq. (9) \((\mu_k = \mu_{kx} + i\mu_{ky} \quad (k = 1, 2))\) are always complex or purely imaginary and are in conjugate pairs as \(\mu_1, \mu_1\) and \(\mu_2, \mu_2\).

In 2D problems, Global Cartesian co-ordinates \((X_1, X_2)\), local Cartesian co-ordinates \((x, y)\) and local Polar co-ordinates \((r, \theta)\) are defined on the crack-tip, as illustrated in Fig. 1, and the complex variable \(z_k\) is defined as: \(z_k = x + \mu_k y_k \quad (k = 1, 2)\).
Displacements and stress fields in the vicinity of the crack-tip were elicited by Sih et al. [4] in terms of mode-I and mode-II stress intensity factors, $K_I$ and $K_{II}$ respectively. The stress components for pure mode-I are [4]

$$\sigma_{11} = \frac{K_I}{\sqrt{2\pi r}} \text{Re} \left[ \frac{\mu_1 \mu_2}{\mu_1 - \mu_2} \left( \frac{\mu_2}{\sqrt{\cos \theta + \mu_2 \sin \theta}} - \frac{\mu_1}{\sqrt{\cos \theta + \mu_1 \sin \theta}} \right) \right]$$

$$\sigma_{22} = \frac{K_I}{\sqrt{2\pi r}} \text{Re} \left[ \frac{1}{\mu_1 - \mu_2} \left( \frac{\mu_1}{\sqrt{\cos \theta + \mu_1 \sin \theta}} - \frac{\mu_2}{\sqrt{\cos \theta + \mu_2 \sin \theta}} \right) \right]$$

$$\sigma_{12} = \frac{K_I}{\sqrt{2\pi r}} \text{Re} \left[ \frac{1}{\mu_1 - \mu_2} \left( \frac{1}{\sqrt{\cos \theta + \mu_1 \sin \theta}} - \frac{1}{\sqrt{\cos \theta + \mu_2 \sin \theta}} \right) \right]$$

and for pure mode-II are [4]

$$\sigma_{11} = \frac{K_{II}}{\sqrt{2\pi r}} \text{Re} \left[ \frac{1}{\mu_1 - \mu_2} \left( \frac{\mu_2^2}{\sqrt{\cos \theta + \mu_2 \sin \theta}} - \frac{\mu_1^2}{\sqrt{\cos \theta + \mu_1 \sin \theta}} \right) \right]$$

$$\sigma_{22} = \frac{K_{II}}{\sqrt{2\pi r}} \text{Re} \left[ \frac{1}{\mu_1 - \mu_2} \left( \frac{1}{\sqrt{\cos \theta + \mu_1 \sin \theta}} - \frac{1}{\sqrt{\cos \theta + \mu_2 \sin \theta}} \right) \right]$$

$$\sigma_{12} = \frac{K_{II}}{\sqrt{2\pi r}} \text{Re} \left[ \frac{1}{\mu_1 - \mu_2} \left( \frac{\mu_1}{\sqrt{\cos \theta + \mu_1 \sin \theta}} - \frac{\mu_2}{\sqrt{\cos \theta + \mu_2 \sin \theta}} \right) \right]$$

The displacements for pure mode-I are [4]

$$u_1 = K_I \sqrt{\frac{2r}{\pi}} \text{Re} \left[ \frac{1}{\mu_1 - \mu_2} \left( \mu_1 p_2 \sqrt{\cos \theta + \mu_2 \sin \theta} - \mu_2 p_1 \sqrt{\cos \theta + \mu_1 \sin \theta} \right) \right]$$

$$u_2 = K_I \sqrt{\frac{2r}{\pi}} \text{Re} \left[ \frac{1}{\mu_1 - \mu_2} \left( \mu_1 q_2 \sqrt{\cos \theta + \mu_2 \sin \theta} - \mu_2 q_1 \sqrt{\cos \theta + \mu_1 \sin \theta} \right) \right]$$

and for pure mode-II are [4]

$$u_1 = K_{II} \sqrt{\frac{2r}{\pi}} \text{Re} \left[ \frac{1}{\mu_1 - \mu_2} \left( p_2 \sqrt{\cos \theta + \mu_2 \sin \theta} - p_1 \sqrt{\cos \theta + \mu_1 \sin \theta} \right) \right]$$

$$u_2 = K_{II} \sqrt{\frac{2r}{\pi}} \text{Re} \left[ \frac{1}{\mu_1 - \mu_2} \left( q_2 \sqrt{\cos \theta + \mu_2 \sin \theta} - q_1 \sqrt{\cos \theta + \mu_1 \sin \theta} \right) \right]$$

where Re denotes the real part of the statement. $p_k$ and $q_k$ are obtained from the roots of characteristic Eq. (9)
In the extended finite element method (XFEM), the finite element approximation is locally enriched to simulate discontinuities and singular fields. The method was first introduced by Belytschko and Black [17], where they partitioned the non-straight crack into some straight segments and mapped them along the direction of the first segment and modeled the mapped crack in the finite element approximation with the framework of partition of unity method (PUM), previously proposed by Melenk and Babuška [32]. By implementing the generalized Heaviside function, proposed by Moës et al. [18], the method was further enhanced, avoiding the need of complicated mapping for arbitrary curved cracks. Some further improvements and extensions can be seen in Sukumar et al. [19], Dolbow et al. [22,33], Belytschko et al. [24], Asadpoure et al. [28–30], while a general description can be found in Mohammadi [31].

In XFEM, the finite element mesh is created regardless of the existence and location of any discontinuity. Then, according to the location of any crack, a few degrees of freedom are added to the selected nodes of original finite element model nearby the discontinuity, and contribute to the approximation through the use of enrichment functions.

### 3.1. Basic equations

Consider a discontinuity in a domain which is represented by a finite element mesh with \( N \) nodes. Belytschko and Black [17] proposed to approximate the displacement field for a point \( x \) inside the domain, based on the XFEM formulation:

\[
\mathbf{u}^e(x) = \sum_{i \in N} \phi_i(x) \mathbf{u}_i + \sum_{j \in N'} \phi_j(x) \psi_j(x) \mathbf{a}_j
\]

where \( n_i \) represents the node \( i \), \( N' \) is the set of nodes that the discontinuity is in its influence (support) domain, \( \phi_i \) is the shape function associated with the node \( i \), \( \psi_j(x) \) is the enrichment function, \( \mathbf{u}_i \) is the FEM vector of regular degrees of freedom and \( \mathbf{a}_j \) is the new additional set of enriched degrees of freedom. The influence domain associated to a node, which is located on an edge, consists of the elements pertaining that node, whereas for an interior node (in higher order elements) is the element enclosing the node (see Fig. 2).

While the first term in Eq. (22) is the conventional finite element approximation to calculate the displacement field, the second term represents the enriched approximation to take into account the existence of discontinuity.

### 3.2. Crack modeling

In order to model crack edges and tips in the extended finite element, Moës et al. [18] suggested that Eq. (22) can be generalized in the following form

\[
p_k = a_{11} \mu_k^2 + a_{12} - a_{16}\mu_k
\]

\[
q_k = a_{12} \mu_k + \frac{a_{22}}{\mu_k} - a_{26}
\]
where $\mathbf{b}_j$ is the vector of additional degrees of freedom which are related to the modeling of crack faces (not crack-tips), $\mathbf{c}_k$ is the vector of additional degrees of freedom for modeling crack-tips, $H(x)$ is the Heaviside function and $F_i(x)$, ($i = 1, 2$) are crack-tip enrichment functions. $\mathbf{N}^f$ is the set of nodes that have crack face (but not crack-tip) in their support domain and $\mathbf{K}_1$ and $\mathbf{K}_2$ are the sets of nodes associated with crack-tip 1 and 2 in their influence domain, respectively.

In Eq. (23), $H(x)$ is the generalized Heaviside function, [18], which becomes +1 if $x$ is above the crack and −1, otherwise. If $x^*$ is the nearest point of a crack to the point $x$ (Fig. 3) and $\mathbf{e}_n$ is the normal vector of the crack alignment in which $\mathbf{e}_s \times \mathbf{e}_n = \mathbf{e}_z$ ($\mathbf{e}_s$ is the unit tangential vector), then

$$H(x) = \begin{cases} +1: & \text{if } (x - x^*)\mathbf{e}_n > 0 \\ -1: & \text{otherwise} \end{cases}$$

Fig. 4 illustrates a finite element mesh for modeling an existing discontinuity. The set of nodes that must be enriched with Heaviside or crack-tip functions are distinguished by circles and triangles, respectively. Other nodes are not affected by the crack and remain well within the classical finite element framework.

### 3.3. Orthotropic crack-tip functions

The following orthotropic crack-tip enrichment functions, recently derived by Asadpoure and Mohammadi [30] are adopted (based on Eqs. (16)–(19)):

$$\{F_i(r, \theta)\}_{i=1}^4 = \left\{ \sqrt{r} \cos \theta_1 \sqrt{g_1(\theta)}, \sqrt{r} \cos \theta_2 \sqrt{g_2(\theta)}, \sqrt{r} \sin \theta_1 \sqrt{g_1(\theta)}, \sqrt{r} \sin \theta_2 \sqrt{g_2(\theta)} \right\}$$

where $g_k(\theta)$ and $\theta_k$ $(k = 1, 2)$ are defined as:

![Fig. 3. Unit tangential and normal vectors for the Heaviside function and nearest point to $x$ on the crack surface, $x^*$.](image)

![Fig. 4. Node selection for enrichment; nodes marked by circles are enriched by the Heaviside function and the triangles are enriched by crack-tip functions.](image)
where $\mu_k = \mu_{kx} + i\mu_{ky}$ are the roots of Eq. (9). Eq. (25) covers all types of composites, as opposed to [28,29], which define the crack-tip enrichment functions for a special type of composites.

3.4. Numerical integration

Numerical integration schemes are used to evaluate various terms of the discretized form of the governing equation in the conventional finite element method. While XFEM utilizes a basically similar approach, however, it requires further improved techniques to account for complexities of stress fields in enriched elements. For instance, ordinary integration techniques are not directly applicable to discontinuous fields in a cracked element, while higher order integration rules should be used for crack-tip elements. The need for partitioning the enriched elements into quadrature sub-elements on both sides of a crack was illustrated by Dolbow et al. [22]. Similar methodology is adopted in this study, where sufficiently accurate results can be obtained by partitioning a finite element domain into 14 and 26 sub-triangles for Heaviside enriched and crack-tip enriched elements, respectively. Further details can be found in [31].

4. Time integration scheme

In this study, the Newmark method is used for the time integration of the extended finite element equations of motion. At a generic time-step $n$, the final discretized simultaneous equations are expressed as:

\[ (M + \beta \Delta t^2 K + \alpha \Delta t C)\dot{u}_n = F - K \left( u_{n-1} + \Delta t \dot{u}_{n-1} + \frac{1 - 2\alpha}{\Delta t^2} \dot{u}_{n-1} \right) - C(\ddot{u}_{n-1} + (1 - \alpha) \Delta t \dot{u}_{n-1}) \]

\[ \ddot{u}_n = \dot{u}_{n-1} + (1 - \alpha) \Delta t \dot{u}_{n-1} + \alpha \Delta t u_n \]

\[ u_n = u_{n-1} + \Delta t \dot{u}_{n-1} + \frac{1 - 2\beta}{\Delta t^2} \dot{u}_{n-1} + \beta \Delta t^2 \ddot{u}_n \]

where $u_n$, $\dot{u}_n$ and $\ddot{u}_n$ are the global vectors of nodal displacements, nodal velocities and nodal accelerations, respectively. $M$, $K$ and $C$ are the mass, stiffness and damping matrices, respectively, and $F$ is the load vector. $\Delta t$ is the time increment at the present time-step and the Newmark’s parameters are chosen to be $\beta = 1/4$ and $\alpha = 1/2$ to fulfill the unconditionally stable criterion.

5. Domain separation integral for orthotropic media

The stress intensity factor is one of the important parameters representing fracture properties of a crack-tip. In the present study, dynamic stress intensity factors, which have been comprehensively described by Freund [34], are evaluated and compared with other available methods to assess the developed XFEM.

The analytical form of dynamic $J$-integral ($J_k^d$), developed by Nishioka and Atluri [35], can be written as:

\[ J_k^d = \int_{F_\Gamma} \left( (W + K)n_k - t_i \frac{\partial q_i}{\partial x_k} \right) d\Gamma + \int_{V_r - V_c} \left( (\rho \ddot{u}_i - f_i) \frac{\partial q_i}{\partial x_k} - \rho \frac{\partial u_i}{\partial x_k} \frac{\partial q_i}{\partial x_k} \right) dA \]

where $u_i$, $t_i$, $f_i$, $n_k$ and $\rho$ denote the displacement, traction, body force, outward direction cosine, and mass density, respectively, and the integral paths $F_\Gamma$, $\Gamma$, and $F_r$ denote near-field, far-field and crack surface paths, respectively. $V_r$ and $V_c$ are the regions surrounded by $F$ and $F_c$ respectively (Fig. 5). $W = (1/2)\sigma_{ij}\epsilon_{ij}$ and $K = (1/2)\rho u_i \ddot{u}_i$ are the strain and kinetic energy densities, respectively. Kim and Paulino [36] proposed an equivalent form of Eq. (31) which is better suited for the finite element method,

\[ J_k^d = \int_{V_r} \left( \alpha_k \frac{\partial q_i}{\partial x_k} \right) dA + \int_{V_r - V_c} \left( (\rho \ddot{u}_i - f_i) \frac{\partial q_i}{\partial x_k} - \rho \frac{\partial u_i}{\partial x_k} \frac{\partial q_i}{\partial x_k} \right) dA \]

where $q$ is a smooth function assumed to vary linearly from $q = 1$ in $V_c$ near a crack-tip to $q = 0$ at the exterior boundary $F$, as depicted in Fig. 6. As a result, the gradient of $q$ vanishes in Eq. (32). The crack-axis (tangential) component ($J_k^\theta$) of the dynamic $J$-integral, which corresponds to the rate of change in the potential energy per unit crack extension ($G$), can be evaluated by the following coordinate transformation:

\[ J_k^\theta = a_k(\theta_0)J_k^d \]
where \( \mathbf{L} \) is the coordinate transformation tensor and \( \theta_0 \) is crack angle. Wu [37] showed that the dynamic energy release rate \( G \) can be related to the instantaneous stress intensity factors for an elasto-dynamically propagating crack with the velocity \( v \). Neglecting \( v \) in the present study of fixed cracks results in:

\[
G = J_0^1 = J_1 \cos \theta_0 + J_2 \sin \theta_0
\]

\[
G = \frac{1}{2} K' L^{-1} K
\]

The nonzero components of \( L \) for orthotropic materials with the symmetry planes coinciding with the coordinate planes has been derived by Dongye and Ting [38]:

\[
L_{33} = \sqrt{C_{55} C_{44}}
\]

\[
\sqrt{C_{66} C_{22} L_{11}} = \sqrt{C_{11} C_{66} L_{22}} = AB^{\frac{1}{2}}
\]

where \( C_{ij} \) are the constitutive coefficients (\( \sigma = C_{ij} \)), and

\[
A = (C_{11} C_{22} - C_{12}^2) C_{66}
\]
\[ B = (C_{66} + \sqrt{C_{11}C_{22}})^2 - (C_{12} + C_{66})^2 \]  

(39)

when ordinary non-singular elements are implemented in the vicinity of a crack-tip, the value of \( J_0 \) may not be evaluated very accurately due to the lack of singularity. On contrary, both the singular and non-singular elements can provide accurate \( K_I \) and \( K_{II} \) values for \( J_0 \).

The component separation method, proposed by Aliabadi et al. [11], is adopted in order to accurately evaluate the in-plane mixed-mode stress intensity factors from the dynamic \( J \)-integral. From Eqs. (13), (14), (18), and (19), Eq. (41) can be derived between the dynamic stress intensity factors and the relative sliding and opening displacements of the crack face:

\[
\delta_I = \frac{D_{11}K_I + D_{12}K_{II}}{D_{11}K_I + D_{12}K_{II}}
\]

(40)

with

\[
D_{11} = \text{Im}\left( \frac{\mu\nu_{p_1-p_2}}{\mu_1-\mu_2} \right), \quad D_{12} = \text{Im}\left( \frac{\mu\nu_{p_1-p_2}}{\mu_1-\mu_2} \right)
\]

(41)

where \( \mu, \nu, p_1 \) and \( q_1 \) are defined in Eq. (9), (20), and (21).

Defining the ratio of opening to sliding displacements by \( Z \),

\[
Z = \frac{\delta_{II}}{\delta_I} = \frac{D_{21}K_I + D_{22}K_{II}}{D_{11}K_I + D_{12}K_{II}}
\]

(42)

the ratio of dynamic stress intensity factors can be obtained as:

\[
H = \frac{K_I}{K_{II}} = \frac{2D_{21} - D_{22}}{D_{21} + 2D_{11}}
\]

(43)

\( K_{II} \) is then determined by substitution for \( K_I \) from Eq. (43):

\[
K_{II} = \sqrt{\frac{2G}{L_1H^2 + L_2}}
\]

(44)

It should be noted that in a general transient analysis, potential oscillations of \( \dot{u} \) may affect the quality of \( J \)-integral calculations through Eq. (32). However, only a limited effect is expected for the present fixed crack analyses. Also, Grégoire et al. [26] have performed an extensive study on stability of dynamic fracture mechanics and conservation of energy. They showed that utilizing the Newmark's implicit mean acceleration approach combined with enriching the cracked elements will provide sufficient accuracy for finding \( u, \dot{u}, \ddot{u} \) and so the \( J \)-integral.

6. Examples

In this section, the following examples are discussed to assess accuracy and performance of the proposed method:

1. Plate with a single central crack with several orientations of the axes of orthotropy.
2. Plate with a single central slanted crack.
3. An inclined center crack in a circular disk subjected to point loads.
4. Beam with a single edge notch.

In all examples, stress intensity factors are obtained with the method discussed in Section 5. Any element cut by a crack is numerically partitioned into five sub-triangles to accurately calculate the numerical integrals as detailed by Dolbow [39].

6.1. Plate with a single central crack

Consider a rectangular plate with a central horizontal crack subjected to a step tensile distributed load. Several orientations of material elastic axes are studied and the plane stress state is presumed (Fig. 7). The size of the cracked plate is \( h = 20 \text{ mm} \) and \( 2a = 4.8 \text{ mm} \). Material properties are: \( E_1 = 118.30 \text{ GPa} \), \( E_2 = 54.80 \text{ GPa} \), \( G_{12} = 8.79 \text{ GPa} \), \( \nu_{12} = 0.083 \) and \( \rho = 1900 \text{ kg/m}^3 \). The time-step is selected from \( \Delta t = h/50c_t \), where \( c_t = \sqrt{C_{22}/\rho} \) is the wave velocity along the \( E_2 \) material-axis. Relative integration domain size, \( r_d/a \), is set to be 0.4.

In order to create the finite element model of the problem, a structured uniform 50 \( \times \) 100 mesh is used with 2 \( \times \) 2 and 6 \( \times \) 6 Gauss quadrature rules for ordinary and enriched elements, respectively (Fig. 8).

Stress intensity factors are calculated by the integral domain. The results show that the maximum values of mode-I stress intensity factor decrease by increasing \( \Phi \) from 0° to 60°. In Figs. 9 and 10, the numerical results are compared by BEM, reported by García-Sánchez et al. [12]; General trends conform well, and the existing differences may be attributed to the fact that these methods have adopted different approaches to calculate dynamic stress intensity factors. The results of the pres-
ent work are smoother than the oscillating BEM results, and its prediction for mode-I dynamic stress intensity factor never becomes negative.

In order to evaluate the effects of different finite element meshes on dynamic stress intensity factors, four different meshes $40 \times 80$, $50 \times 100$, $60 \times 120$ and $70 \times 140$ are utilized. The material angle is considered zero and the time-step is

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**Fig. 7.** Geometry and loadings of a single central crack in a rectangular plate with several orientations of the axes of orthotropy.

**Fig. 8.** The finite element model for simulation of a single central crack in a rectangular plate with several orientations of the axes of orthotropy: (a) whole view of FEM mesh; (b) Enlarged view of around the crack-tip, with the dynamic $J$-integral contour (dash line) and the effected elements represented by star points.
set to $\Delta t = h/40c_L$. According to Fig. 11, the results are closely matched with maximum differences of 7.1%. It shows that even with a coarse finite element mesh accurate solutions can be obtained from the XFEM modeling. Fig. 12 illustrates the $\sigma_2$ stress contour with a clear indication of stress concentration around the crack-tips. The maximum concentration factor is predicted to be around 6.6.

Fig. 9. Normalized mode-I dynamic SIFs for a single central crack in a rectangular plate with several orientations of the axes of orthotropy.

Fig. 10. Normalized mode-II dynamic SIFs for a single central crack in a rectangular plate with several orientations of the axes of orthotropy.
Fig. 11. Normalized mode-I dynamic SIFs for a horizontal central crack in a rectangular plate for different meshes.

Fig. 12. $\sigma_2$ stress contour for a horizontal central crack in a rectangular plate at time $c_L t/h = 2.2$. 
6.2. Plate with an edge crack

In this example, a 40 × 52 mm tensile plate with a 12 mm edge crack is considered (Fig. 13). The plane stress condition is considered and the material properties are: \( E_1 = 118.30 \text{ GPa} \), \( E_2 = 54.80 \text{ GPa} \), \( G_{12} = 8.79 \text{ GPa} \), \( \nu_{12} = 0.083 \) and \( \rho = 1900 \text{ kg/m}^3 \). Relative integration domain size, \( r_d/a \), is set to be 0.4. Two different time-steps \( \Delta t = a/3c_L \) and \( \Delta t = a/10c_L \) are selected to evaluate the effect of different time-steps.

A 78 × 60 finite element mesh is used for discretization of the model with 2 × 2 and 5 × 5 Gauss quadrature rules for ordinary and enriched elements, respectively (Fig. 14).

The results of XFEM analysis for the mode-I stress intensity factor are depicted in Fig. 15 and compared with the reference BEM and the conventional FEM results (via ANSYS), reported by García-Sánchez et al. [12]. Very good agreement is clearly observed. Fig. 15 also compares the results for different time-steps. Again, similar results are obtained.

The rate of convergence of the stress intensity factor for various relative integration domain sizes, \( r_d/a \), are depicted in Fig. 16. In comparison to BEM results, it is concluded that the results are more accurate when \( r_d/a \) is between 0.2 and 0.6. For higher values of \( r_d/a \), the results show a different trend for mode-I stress intensity factor in earlier times. It can be interpreted as a sign of improper size of the integration domain with respect to the crack length.

In order to compare and assess the effects of different number of Gauss quadrature points, five different Gauss quadrature rules \((2 \times 2–4 \times 4), (2 \times 2–7 \times 7), (4 \times 4–4 \times 4), (4 \times 4–7 \times 7), (5 \times 5–7 \times 7)\) for (ordinary – enriched) elements are used to calculate the mode-I stress intensity factor based on the 78 × 60 finite element mesh. According to Fig. 17, the results are...
virtually the same, with the maximum difference of about 3.03%. Practically, the results show that even with the minimum number of Gauss points, accurate results can be obtained.

In addition, to determine the effects of different meshes on dynamic stress intensity factors, four different meshes $50 \times 42$, $78 \times 60$, $100 \times 84$ and $125 \times 105$ are used to simulate the problem (Fig. 18). Again, similar results are obtained with the maximum difference of 4.3%. Therefore, XFEM can accurately predict the results with a relatively coarse mesh of $50 \times 42$ elements.

6.3. An inclined center crack in a disk subjected to point loads

Consider a disk with an inclined center crack is subjected to double point loads with the geometry and boundary conditions as illustrated in Fig. 19. The size of the cracked disk is $R = 10$ and $a = 2$. The material elastic axes are supposed to be
coincident with $x_1$ and $x_2$-axes and the following material properties are utilized: $E_{11} = 0.1$, $E_{22} = 1.0$, $G_{12} = 0.5$, $v_{12} = 0.03$, $q = 1200/C_2$. The analysis is performed by $\Delta t = a/3c_L$ time-step. Relative integration domain size, $r_d/a$, is equal to 0.4. The static solution was previously studied by Asadpoure et al. [28], which is used to compare and discuss the results. Also, Damping is superimposed in order to achieve a faster convergence to the static solution. The finite element model adopted in this example has 852 four-node elements and 877 nodes, depicted in Fig. 20, with $2 \times 2$ and $5 \times 5$ Gauss quadrature rules for ordinary and enriched elements, respectively.

Because no reference analytical solution or numerical results were available for dynamic loadings, the present results have been compared with the static results proposed by Asadpoure et al. [28]. The results for both stress intensity factors clearly illustrate that the dynamic solution oscillates around the reference static results (Figs. 21 and 22).
results, the mode-I stress intensity factor reduces with the increase of orthotropy angle from $U = 0^{\circ}$ to $U = 45^{\circ}$. As expected, the mode-II stress intensity factor increases in this span. Furthermore, the period of variations of the stress intensity factors; the time between two corresponding peaks along the $E_1$ material-axis, can be anticipated by $T = 2R/c_R$ where $c_R = \sqrt{C_{11}/\rho}$; resulting in 0.069 s compared with 0.056 from the XFEM solution. Fig. 23 illustrates the distribution of $\sigma_2$ stress component over the circular plane, with a clear indication of stress concentration around the crack-tip.

6.4. Three-point bending beam

For the final example, a three-point bending test on an isotropic beam with a fixed crack, subjected to a step loading, is considered (as shown in Fig. 24). Geometry and material properties are define as: Crack length $a = 0.005$ m, height $W = 0.01$ m, span $S = 0.04$ m, length $L = 0.055$ m, depth $B = 1.0$ m, Young’s modulus $E = 200$ GPa, material density $\rho = 7860$ kg/m$^3$, Possion’s ratio $\nu = 0.3$ and the pressure $\sigma = 500$ N/m is applied at the top of the beam in the span of $l = 0.02$ m.

The analytical solution was obtained by Kishimoto et al. [40] for the mode-I stress intensity factor in static state,

$$K_{IS} = \frac{6BSL}{4W^2} \sqrt{\pi a \psi\left(\frac{a}{W}\right)}$$

(45)
Fig. 21. Variation of the mode-I stress intensity factor when crack inclination varies with respect to \( x_1 \)-axis from 0° to 45°.

Fig. 22. Variation of the mode-II stress intensity factor when crack inclination varies with respect to \( x_1 \)-axis from 0° to 45°.
Fig. 23. $\sigma_2$ stress contour for an inclined center crack in a disk subjected to point loads at time $c_l t/D = 20$ for the crack inclination of 45°.

Fig. 24. Geometry and loading of a plane strain beam with an edge crack.

Fig. 25. Mode-I dynamic SIFs for a plane strain isotropic beam with an edge crack.
where $\psi$ is defined as

$$
\psi \left( \frac{a}{W} \right) = 1.090 - 1.735 \left( \frac{a}{W} \right) + 8.20 \left( \frac{a}{W} \right)^2 - 14.18 \left( \frac{a}{W} \right)^3 + 14.57 \left( \frac{a}{W} \right)^4
$$

(46)

A $110 \times 20$ finite element mesh is used with the $2 \times 2$ and $6 \times 6$ Gauss quadrature rules for ordinary and enriched elements, respectively. The relative integration domain size, $r_d/a$, is considered to be 0.35. The time-step is set to $\Delta t = 5 \mu s$ and the total time of simulation is 265 $\mu s$. The dynamic stress intensity factor, predicted by the present XFEM approach, oscillates about its analytical static value, as depicted in Fig. 25.

![Fig. 25.](image)

**Fig. 25.** Variation of mode-I dynamic SIF for a plane strain orthotropic beam with an edge crack with respect to different meshes.

![Fig. 26.](image)

**Fig. 26.** Effect of damping on mode-I dynamic SIF for a plane strain orthotropic beam with an edge crack.

![Fig. 27.](image)

**Fig. 27.** Variation of mode-I dynamic SIF for a plane strain orthotropic beam with an edge crack with respect to different meshes.
Now, the beam is considered as an orthotropic beam with the following material properties: $E_1 = 1000$ GPa, $E_2 = 100$ GPa, $E_3 = 80$ GPa, $G_{12} = 70$ GPa, $\nu_{12} = 0.38$, $\nu_{13} = 0.49$, $\nu_{23} = 0.41$ and $\rho = 1200$ kg/m$^3$. No results are available in literature for such a complex solution of dynamic orthotropic stress intensity factor. A $145 \times 29$ finite element mesh and the time-step $\Delta t = 1\mu s$ are used for the analysis. The relative integration domain size, $r_d/a$, is considered to be 0.35. The results for different damping ratios are depicted in Fig. 26. To include the effects of damping, the following linear stiffness-based damping is used:

$$C = \alpha K$$

where $\alpha$ is a damping coefficient. These results show a good convergence to a static orthotropic solution in time. Also, different finite element meshes are used to discuss the effect of coarse and fine meshes on the results. According to Fig. 27, the

![Image of deformation in different time-steps](image-url)

Fig. 28. Deformation of the plane strain orthotropic beam with an edge crack in different time-steps.
maximum difference between the results is nearly 2.7%, which indicates the mesh independency of the results. In Fig. 28, the exaggerated deformation of beam is illustrated. It is observed that when the dynamic stress intensity factor reaches to its maximum at $t = 13$ μs, the crack opening displacement becomes maximum too, and the crack starts to close afterwards.

7. Conclusion

Dynamic analysis of fixed cracks in orthotropic media was studied in this paper. The extended finite element method was adopted for modeling the crack and analyzing the domain numerically. Orthotropic crack-tip (near-tip) enrichment functions, which can be applied to all types of orthotropic materials, are implemented. Mixed-mode stress intensity factors (SIFs) were determined based on evaluation of the dynamic $J$-integral ($J_k$), to determine fracture properties of the domain. The results, obtained by the proposed method, are in good agreement with other numerical or (semi-) analytical methods. Moreover, numerical results show that the proposed method is not sensitive to the effects of different time-steps or Gauss points. Also, the values of dynamic $J$-integral and consequently stress intensity factors become independent from the domain size, if the relative integration domain size remains within the range of 0.2–0.6.

References

