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2. Figures 1, 2, 3, 4, 7, 10, 12, 13, 16, 17, 18, 20, 23, 25, 27, 28, 30 are poor quality.  
3. Please update Ref. [42].  
4. Please check the renumbering of Appendix equations.
Fracture analysis of FRP-reinforced beams by orthotropic XFEM

S. Esna Ashari and S. Mohammadi

Abstract
The extended finite element method is adopted for fracture analysis of delamination problems in fiber-reinforced polymer (FRP) reinforced beams. In this method, the stress singularities near the debonding crack tip are modeled by newly proposed orthotropic bimaterial enrichment functions, while crack faces and the material interface simulated by the Heaviside function and the weak discontinuity enrichment function, respectively. The orthotropic crack tip enrichments are utilized in orthotropic FRP plate and in a reduced isotropic form in the beam. The strain energy release rate is numerically evaluated using mixed-mode stress intensity factors which are evaluated by the means of domain interaction integral approach. The accuracy of this approach is examined through some numerical examples and the results are compared with available reference results.

Keywords
extended finite element method, delamination, debonding, fiber-reinforced materials, orthotropic enrichment functions, strengthened beams

Introduction
Fiber-reinforced polymers (FRPs) are widely used in many kinds of engineering applications such as structural strengthening, seismic retrofitting, and repair applications. FRP materials, especially in the form of laminates and pultruded plates, are becoming a material of choice in order to strengthen existing concrete or steel structures due to their high strength and stiffness-to-weight ratio, resistance to corrosion and environmental effects, and substantial fatigue durability.

As a flexural reinforcement for beams, FRP plates are externally bonded to the tension face of a concrete or steel beam, by means of an adhesive layer, which significantly improves the stiffness and load capacity of beams. The performance of the FRP-beam interface in providing an effective mean for stress transfer is so important and the success of the strengthening procedure depends mainly on the quality of the FRP-beam joint, the efficiency of the adhesive, and its vulnerability to debonding failures. Debonding failures are usually caused by high concentrations of normal and shear stresses at the end of the bonded plate, bonding defects in the application of the FRP plate, or the effect of impact loadings. Debonding failures are associated with the initiation and growth of horizontal cracks at or very close to the adhesive-beam interface. Any kind of debonding failure may reduce the expected lifetime extension and significantly weaken the structural integrity and may also damage the structural performance and safety due to reduced ductility. In many cases, the debonding causes a brittle and sudden fracture mechanism of the component.

There are generally two approaches to the simulation of debonding failures in FRP-strengthened structures, recognized as stress- and fracture mechanics-based approaches. The first one involves evaluation of interfacial stress distribution in strengthened members based on an elastic assumption and comparison of the computed stresses with the adhesive strength to predict the load level of debonding failure. As an example,
Roberts\textsuperscript{15} presented a general formulation for the analysis of composite beams with partial interaction and proposed some models for the interface shear stress concentration near the ends of the epoxy-bonded external plates. Taljsten\textsuperscript{16} used the linear elastic theory, derived for shear and peeling stresses, to calculate the critical stress levels at the end of an outer reinforcement plate.

The second approach refers to fracture mechanics and is more appropriate to predict debonding failures due to the presence of stress singularities at the end of plate/beam interface and the crack nature of the debonding failures. Within the category of fracture mechanics models, two main approaches are recognized. The first approach is based on cohesive interface modeling and the second approach on linear elastic fracture mechanics (LEFM) crack growth simulation. The cohesive interface approach employs a layer of interface elements or a general contact interface between the FRP and the beam in which debonding is simulated as failure of the interface elements.\textsuperscript{12,13,17,22}

In the second approach, fracture mechanics parameters, typically the energy release rate (ERR) or the $J$ integral, are evaluated analytically or numerically and compared with the relevant critical fracture energy. Based on this concept, the initiation and propagation of the horizontal edge delamination crack occur if the amount of energy released by the system due to infinitesimal crack growth is larger than the specific fracture energy. Existing LEFM debonding analyses mainly differ in the analytical and computational tools adopted for the evaluation of fracture mechanics parameters. For example, Yang et al.\textsuperscript{23} adopted the finite element method (FEM) and the virtual crack extension method while Greco et al.\textsuperscript{24} used a layered shear deformation model for the stress analysis and the virtual crack closure method for the assessment of ERR. Rabinovitch and Frostig\textsuperscript{25} used a high-order theory for the stress analysis of the strengthened beam and the $J$ integral formulation for evaluation of ERR. An alternative approach, based on simplified stress analysis tools and assessment of ERR through numerical differentiation of the total potential energy, was proposed by Rabinovitch.\textsuperscript{26}

The common numerical method used for analyzing these kinds of problems is the FEM, which is capable of analysis of non-linear systems and can be easily adapted to general boundary conditions and complex geometries. Besides the advantages of the FEM, this method is unable to represent the exact singular stress fields near the crack tip. Moreover, in crack problems such as debonding analysis, the elements associated with cracks must conform to crack faces and remeshing techniques may be required to simulate the crack propagation.

The extended finite element method (XFEM), enhances the conventional FEM capabilities by avoiding the requirement of mesh to conform to weak or strong discontinuities and the necessity of adaptive remeshing techniques in crack growth problems. This method was first proposed by Belytschko and Black,\textsuperscript{31} combining the FEM with the concept of partition of unity. In XFEM, elements containing a crack are enriched with a discontinuous function and near-tip asymptotic displacement fields. Sukumar et al.\textsuperscript{32} developed partition of unity-based enrichment techniques for bimaterial interface cracks between two isotropic media, while Remmers et al.\textsuperscript{33} presented a partition of unity finite element (FE) based on a solid-like shell element for simulation of delamination growth in thin layered structures. Nagashima and Suemasu\textsuperscript{34,35} applied XFEM to stress analyses of structures containing interface cracks between dissimilar isotropic materials and also modeled interlaminar delaminations of composite laminates by the near-tip functions for homogeneous isotropic cracks in order to examine the behavior of orthotropic composite materials.

Asadpoure et al.,\textsuperscript{36,37} Asadpoure and Mohammadi,\textsuperscript{38} and Mohammadi\textsuperscript{39} extended the method to orthotropic media by deriving a new set of orthotropic enrichment functions. Later, Motamedi and Mohammadi\textsuperscript{40,41} further extended the method to dynamic analysis of stationary and propagating cracks in orthotropic media and proposed new time-independent moving orthotropic enrichment functions for dynamic analysis of composites.\textsuperscript{42} Recently, Esna Ashari and Mohammadi\textsuperscript{43–45} proposed new orthotropic interlaminar crack tip enrichment functions, which extend the capability of XFEM to full fracture analysis of two-dimensional crack problems in isotropic and orthotropic multilayer media. The interlaminar crack enrichment functions were derived from analytical asymptotic displacement fields around a traction-free interfacial crack.

In this research, XFEM is adopted for fracture analysis of debonding problems in FRP-strengthened beams by means of recently proposed bimaterial orthotropic enrichment functions.\textsuperscript{45} The near-tip enrichment functions represent the stress singularities near the debonding crack tip. In all numerical simulations, the beams are assumed as an isotropic material, while the FRP plate as an orthotropic material. Combined modes I and II loading conditions are studied and the strain ERR is numerically evaluated using the mixed-mode stress intensity factors (SIFs). SIFs are determined by the use of domain form of the contour interaction integral. In order to examine the performance of the proposed approach, various numerical examples are simulated and the results compared with available reference solutions.
Mechanics of bimaterial interface cracks

For modeling an interface crack between FRP (orthotropic material) and concrete or metal (isotropic material) in the XFEM, proper enrichments, extracted from the near crack tip displacement fields, are required. Here, the analytical near crack tip displacement fields of a traction-free crack between two orthotropic materials, derived by Lee,\textsuperscript{46} are used. These displacement fields can be utilized in both orthotropic and isotropic materials.

Figure 1 indicates a traction-free interface crack located in \((x,y)\) plane between two orthotropic materials. The elastic principal directions are assumed normal to or parallel with the crack direction and the local polar coordinates \((r,\theta)\) defined on the crack tip.

For each orthotropic material, a partial differential equation in terms of a complex variable \(z = x + my\) can be obtained with the following characteristic equation:\textsuperscript{47}

\[
m^4 + 2B_{12}m^2 + K_{66} = 0
\]

where

\[
B_{12} = \frac{1}{2} \left( \frac{2a_{12} + a_{66}}{a_{11}} \right), \quad K_{66} = \frac{a_{22}}{a_{11}}
\]

\(a_{ij}(i,j = 1,2,6)\) are the material parameters used in the stress-strain law:\textsuperscript{48}

\[
\varepsilon_{\alpha} = a_{\alpha \beta} \sigma_{\beta} \quad (\alpha, \beta = 1,2,6)
\]

The characteristic roots of Equation (1) \((m_1\) and \(m_2)\) in most orthotropic materials, known as type 1, are purely imaginary:

\[
m_1 = ip, \quad m_2 = iq
\]

where

\[
p = \sqrt{B_{12} - B_{12}^2 - K_{66}}
\]

\[
q = \sqrt{B_{12} + B_{12}^2 - K_{66}}
\]

The asymptotic displacement fields in the vicinity of the orthotropic biomaterial crack tip can be defined in terms of \(p, q\) and the polar coordinate system \((r,\theta)\).\textsuperscript{46} They are described comprehensively in Appendix.

The functions that define the nature of the analytical displacement fields will be added to the FEM.
approximation as the XFEM enrichment functions in order to represent the singular crack tip stress fields.

In an isotropic material, values of \(B_{12}\) and \(K_{06}\) are equal to 1 and Equation (1) is reduced to:

\[ m^4 + 2m^2 + 1 = 0 \]  

(7)

So, the roots are:

\[ m_1 = i, m_2 = i \]  

(8)

and

\[ p = q = 1 \]  

(9)

Consequently, all the displacement and stress fields and the enrichment functions are simplified in the isotropic material.

**Extended finite element method**

The XFEM is a numerical method which allows for the modeling the arbitrary strong and weak discontinuities, independent of the mesh discretization. XFEM uses the partition of unity property to define an enriched solution in order to reproduce discontinuous or singular fields. In this method, the mesh does not require to conform to the discontinuities nor significant mesh refinement is necessary near the singularities. A single XFEM element, without the loss of accuracy, may contain two different materials and an interface in between.

**XFEM for interface crack modeling**

For an interface crack located in a domain within an FE mesh, shown in Figure 2, the XFEM displacement approximation around the crack, \(u^a\), is enriched with the discontinuous Heaviside function to model crack surfaces, the crack tip asymptotic displacement fields to model crack tips, and the weak discontinuity enrichment function used to model the interface of two materials. It can be expressed in the following form:

\[
\begin{align*}
    u^a(x) &= \sum_I N_I(x)u_I + \sum_J a_J N_J(x)(H(x) - H(x_J)) \\
    &+ \sum_K N_K(x)\left(\sum_I b_I^e(F_I(x_K) - F_I(x_K))\right) \\
    &+ \sum_r c_r N_r(x)\phi_r(x) \\
    &+ \sum_{n_r} \bar{N}_{n_r}(x)(H(x) - H(x_{n_r}))
\end{align*}
\]  

(10)

The first term is the classical FE approximation where \(N_I\) is the FE shape function and \(u_I\) the vector of regular degrees of nodal freedom.

The other terms are the enriched approximation, where \(N_I^H\), \(N_F^r\), and \(N_F^x\) stand for the set of nodes that have crack face, crack tip, and weak discontinuity in their support domain, respectively. \(a_J, b_I^e\), and \(c_r\) denote the vectors of additional degrees of nodal freedom for modeling crack faces, crack tips, and weak discontinuity interfaces, respectively.

**Crack face enrichment**

\(H(x)\) is the Heaviside function that simulates the discontinuity of displacement within an FE at the crack location:

\[
H(x) = \begin{cases} 
  +1 & \text{if } \phi(x) > 0 \\
  -1 & \text{if } \phi(x) < 0 
\end{cases}
\]

(11)

where \(\phi(x)\) is the signed distance function computed in a normal direction to the crack side and takes the value +1 if the point \(x\) is above the crack and –1, otherwise.\(^39\)

**Material interface enrichment**

\(\chi(x)\) is the enrichment function used for modeling weak (material) discontinuities within an FE and is defined in terms of the signed distance function as:

\[
\chi(x) = |\phi(x)|
\]

(11)

This enrichment is used where a continuous displacement field exists but the strain fields are discontinuous, such as the interface of two different materials. This enrichment function is used only for the elements that are cut by the interface, avoiding generation of excessive errors in surrounding elements.\(^45\) If the mesh conforms to the material interface, no nodes are enriched by the weak discontinuity function.

On an element cut by the crack or interface, the integration of XFEM stiffness matrix and force vector needs to be performed carefully. An efficient way for this procedure is to divide the element into sub-domains matching the crack faces or material interface. Then, the number of integration points over each sub-domain is chosen accordingly.\(^39\)

**Crack tip enrichment functions**

The following basic crack tip enrichment functions that span the crack tip asymptotic displacement field
\[ \{F(r, \theta)\}_{l=1}^{8} = \left[ e^{-\theta_0 \cos(\epsilon \ln(r) + \frac{\theta_0}{2})\sqrt{r_2}} e^{-\theta_0 \sin(\epsilon \ln(r) + \frac{\theta_0}{2})\sqrt{r_2}}, \right. \]
\[ e^{-\theta_0 \cos(\epsilon \ln(r) - \frac{\theta_0}{2})\sqrt{r_2}}, e^{-\theta_0 \sin(\epsilon \ln(r) - \frac{\theta_0}{2})\sqrt{r_2}}, \]
\[ e^{-\theta_0 \cos(\epsilon \ln(r) + \frac{\theta_0}{2})\sqrt{r_2}}, e^{-\theta_0 \sin(\epsilon \ln(r) + \frac{\theta_0}{2})\sqrt{r_2}}, \]
\[ e^{-\theta_0 \cos(\epsilon \ln(r) - \frac{\theta_0}{2})\sqrt{r_2}}, e^{-\theta_0 \sin(\epsilon \ln(r) - \frac{\theta_0}{2})\sqrt{r_2}} \] \] (12)

with
\[ r_j = r_j \sqrt{\cos^2 \theta + Z_j^2 \sin^2 \theta}, \quad j = l, s, \]
\[ \theta_j = \tan^{-1}(Z_j \tan \theta) \] (13)

where \( \epsilon \) (the oscillation index) and all parameters have been defined in Appendix. It should be noted that in the isotropic material \( r_j = r \) and \( \theta_j = \theta \).

It is worth noting that these set of functions contain combined typical terms of \( \sqrt{r_2} \cos(\theta_j/2) \) or \( \sqrt{r_2} \cos^2 \theta_j + Z^2 \sin^2 \theta_j \) \( \cos(\theta_j/2) \). These include all generalized terms of \( \sin \theta_j, \sin \theta_j \sin(\theta_j/2), \sin \theta_j \cos(\theta_j/2), \ldots \) and may generally span all various components of orthotropic material and isotropic bi-material enrichment functions, as proposed by Sukumar.\(^{32,45}\)

**Evaluation of ERR**

The value of ERR is computed from the SIFs:\(^{49}\)
\[ G = \frac{1}{4 \cosh(\pi \epsilon)} \left[ D_{22} K_1^2 + D_{11} K_2^2 + 2 D_{12} K_1^2 K_2^2 \right] \] (14)

\[ D = \left[ L_{11}^{-1} + L_{22}^{-1} \right], \] where \( L \) is the Barnett and Lothe tensor, defined in Equation (20).\(^{30}\) \( K_1 \) and \( K_2 \) are the modes I and II SIFs. These are evaluated using the domain integral method, developed by Chow et al.\(^{51}\)

Evaluation of different SIFs, in general, mixed-mode loadings is performed by the interaction integral \( M \),
\[ M = \int_{A} \left[ \sigma_{i j} u_{i j}^{\text{aux}} + \sigma_{i j} u_{i j}^{\text{aux}} - W^{(1, 2)} \delta_{ij} \right] \eta_{j} dA \] (15)

where superscript aux stands for the auxiliary state, \( A \) the region including the crack tip confined by \( \Gamma \) (Figure 3), and \( q \) a smooth function that is unity inside the domain \( A \) and zero outside the domain. Therefore, the integrand of (15) vanishes for all the elements that lie completely inside the domain (because of constant \( q \) and zero \( q_j \)), and it can simply be evaluated over the elements that are cut by the contour \( \Gamma \) (non-zero \( q_j \)). \( W^{(1, 2)} \) for linear elastic condition is defined as
\[ W^{(1, 2)} = \frac{1}{2} \left( \sigma_{i j}^{\text{aux}} + \sigma_{i j}^{\text{aux}} \right) \] (16)

In the interaction integral method, auxiliary fields are superimposed onto the actual fields to satisfy equilibrium equation and traction-free boundary condition on crack surfaces.\(^{52}\) One of the choices for the auxiliary state is the displacement and stress fields in the vicinity of the interfacial crack tip.\(^{46}\)

The SIFs are then determined from the \( M \) integral:\(^{51}\)
\[ \kappa_i = 2 \sum_{m=1}^{2} U_{i m} M \left[ u_{i} \eta_{m}^{\text{aux}}(m) \right] \] (17)

where \( u_{i}^{\text{aux}(1)} \) and \( u_{i}^{\text{aux}(2)} \) are the auxiliary states associated with \( K_1 = 1, K_2 = 0 \) and \( K_1 = 0, K_2 = 1 \), respectively, and
\[ U = \left[ \left( L_{11}^{-1} + L_{22}^{-1} \right)^{-1} \left[ I + \left( L_{11}^{-1} + L_{22}^{-1} \right)^{-1} \right] \right]^{-1} \] (18)

Components of \( S \) and \( L \) are computed for each individual layer, denoted by \#1 and \#2 for upper and lower materials, respectively. Explicit expressions of \( S \) and \( L \) for an orthotropic material have been derived by Dongye and Ting\(^{33}\) and Deng.\(^{54}\)

\[ S_{21} = \frac{C_{12} \left( \sqrt{C_{11} C_{22} - C_{12}} \right)}{C_{22} (C_{12} + 2 C_{66} + \sqrt{C_{11} C_{22}})} \] (19)

**Figure 3.** The contour \( \Gamma \) containing the crack tip.
and $C_{ij}$ is the contracted form of the fourth-order elastic constant tensor.

**Numerical examples**

In all numerical examples, the ERR is obtained with the method mentioned in the previous section. For the integration purposes, a $2 \times 2$ Gauss quadrature rule is used over regular elements, while the quadrature partitioning approach adopted over enriched elements. Any element cut by the crack faces or the material interface is divided into four sub-triangles and the elements containing the crack tip are partitioned into six sub-triangles. Seven Gauss points are utilized for integrations in each sub-triangle. Elements next to the main tip-enriched element need not any sub-division, and a simple $5 \times 5$ Gauss quadrature rule is utilized.

**Concrete beams strengthened with externally bonded glass fiber-reinforced polymer**

In this example, the flexural behavior of a reinforced concrete beam strengthened by epoxy-bonded glass-fiber reinforced plate is investigated by the XFEM. The concrete beam is under the four-point bending test while reinforced by glass fiber-reinforced polymer (GFRP) bonded to the tension face, as shown in Figure 4. The plane stress state is assumed and the material properties are defined as:

**Concrete.**

\[
E = 21,000 \text{ MPa}, \quad v = 0.2.
\]

**GFRP.**

\[
E_1 = 37,230 \text{ MPa}, \quad E_2 = 1861 \text{ MPa},
\]

\[
G_{12} = 1618 \text{ MPa}, \quad \nu_{12} = 0.27.
\]

The beam dimensions are:

\[
b = 0.2 \text{ m}, \quad h = 0.3 \text{ m}, \quad L = 3 \text{ m}
\]

Two different thicknesses of the GFRP plate ($t_p = 2.4 \text{ mm}$) are considered.

A uniform structured FE model with 1386 quadrilateral four-noded elements and 1500 nodes is employed. Figure 5 depicts the FEM mesh. It should be noted that only one row of elements contains the GFRP laminate and part of the concrete at once. These elements are enriched by the weak discontinuity function.

The analysis is performed for different applied loadings within the elastic condition and the mid-span deflection is measured in each case and the results are compared with the results reported in Hsu study. Hsu employed non-linear models of the FE code ABAQUS for simulating the behavior of strengthened concrete beams.

The variation of mid-span deflection as a function of applied load in elastic zone is plotted in Figure 6 for different GFRP plate thicknesses in comparison with the reference results. A very close agreement is observed between XFEM and reference curves in the linear elastic part, indicating the accuracy of XFEM approach in modeling FRP, part of concrete, and the material interface in a single element.

**FRP-reinforced concrete cantilever beam subjected to an edge moment**

In this example, analysis of an edge debonding problem in concrete beams strengthened/repaired with
externally bonded composite laminated plates, as shown in Figure 7, is considered. A bilayer specimen composed of two homogeneous elastic layers with an interface crack is simulated by present XFEM and the results are compared with those obtained by Greco et al.\textsuperscript{24} The first layer is isotropic and represents the concrete beam to be strengthened and the other one the orthotropic layer of typical carbon–epoxy unidirectional lamina. The beam is subjected to an edge moment which is modeled by means of two opposite uniform distributions of normal pressure acting at the $x = L_c$ edge with the plane stress condition. The material properties are defined as:

**Concrete.**

$$E = 30,000 \text{ MPa}, \ G_{12} = 12,820.53 \text{ MPa}.$$  

**CFRP.**

$$E_1 = 160,000 \text{ MPa}, \ E_2 = 11,860 \text{ MPa}, \ G_{12} = 5333.3 \text{ MPa}, \ \nu_{12} = 0.3.$$  

The dimensions of the problem are:

$$L_c = 1500 \text{ mm}, \ L_p = 1200 \text{ mm}, \ t_c = 300 \text{ mm}, \ t_p = 4 \text{ mm}.$$  

In order to evaluate the value of ERR, adaptive structured FE models with different numbers of quadrilateral four-noded elements are employed for different crack lengths. Element sizes are substantially smaller in the vicinity of the crack, and the tip element sizes in $x$- and $y$-directions are 4 and 1.07 mm, respectively, while the FRP laminate is discretized by 3.5 elements in $y$-direction; so, the top element includes FRP, concrete, and interface. The total number of rows and columns of FEs are 82 and 134, respectively, resulting in 6834 elements and 7020 nodes. Figure 8 illustrates the typical mesh utilized for $a = 200$ mm.

The value of the relative $J$-integration domain size, $r_d$ (the distance between the crack tip and each edge of the integration domain), is assumed to be different for CFRP strip below the crack edge (about the thickness of FRP plate) and about 27 mm for other three edges (Figure 9) in the present XFEM analysis.

Greco et al.\textsuperscript{24} evaluated the strain ERR from a proposed analytical method and an FEM using a mesh characterized with the crack tip element size of 1 mm in the $x$-direction. The FE mesh, obtained by the commercial FE code LUSAS, is illustrated in Figure 10, where the mesh refinement in a region enclosing the crack tip is also shown. It should be noted that the adhesive layer was modeled and delamination assumed
Figure 7. The geometry and boundary conditions of an interface crack in FRP-plated cantilever beam under bending.

Figure 8. FEM discretization of FRP-reinforced cantilever beam.

Figure 9. The rectangular $J$ integral domain.
to occur at the interface between beam and the adhesive layer. The FEs were 1 mm long and 1 mm high in the FRP and adhesive layers, and 160 elements were used across the thickness of concrete beam; including 80 elements with a constant height in the upper half of the vertical section, and a graded mesh starting with an aspect ratio of 1 near the crack tip in the lower half. In the rest of the model, a uniform mesh was utilized. The total number of elements and nodes in Greco et al.\textsuperscript{24} model were about 40,720 and 40,979, respectively.

The results of strain ERRs as a function of the crack length, obtained by XFEM, are depicted in Figure 11 and compared with reference results.\textsuperscript{24} This figure indicates that the total ERR remains practically constant for most of the crack lengths and then decreases toward 0 rapidly as the crack reaches the beam end.

Very close agreements are observed between the XFEM and available reference analytical and numerical results, while the number of elements in the XFEM mesh is about 1/6th the number of elements of mesh utilized in Greco et al.\textsuperscript{24} It should be noted that the
XFEM results are closer to the analytical results than the reference two-layer FE model as the crack reaches the beam end.

**FRP-reinforced cantilever beam subjected to an edge transverse force**

A cantilever beam specimen subjected to an edge transverse force, as illustrated in Figure 12, is considered. Again, the first layer is (isotropic) concrete and the second layer the (orthotropic) CFRP plate used to strengthen the concrete beam. The geometry of the specimen is the same as previous problem and the material properties are defined as:

**Concrete.**

\[ E = 30,000 \text{ MPa}, \ G_{12} = 12,820.53 \text{ MPa}. \]
CFRP strips.

\[ E_1 = 160,000 \text{ MPa}, \quad E_2 = 7610 \text{ MPa}, \]
\[ G_{12} = 5333.3 \text{ MPa}, \quad v_{12} = 0.3. \]

The analysis is performed for the bimaterial plate in a plane stress state, subjected to the edge transverse force and for a range of crack lengths. Again, adaptive structured FE models similar to example 2 are used for different crack lengths and the solution around the crack tip is enhanced with new orthotropic enrichment functions. The mesh discretization used for \( a = 800 \text{ mm} \) is illustrated in Figure 13. The FRP laminate is discretized into 4.5 elements in \( y \)-direction and the tip element sizes in \( x \)- and \( y \)-directions are 4 and 0.91 mm, respectively, with the total number of 8308 elements and 8505 nodes.

The same relative \( J \) integral domain size as example 2 is used to evaluate the strain ERRs. The variation of strain ERRs with respect to the crack length is depicted in Figure 14.

The results of XFEM simulations are compared with the results reported by Greco et al.\textsuperscript{24} using an analytical formulation and an FE model (similar to Figure 10) in Figure 14.

Clearly, very accurate results are obtained by new XFEM formulation. Similar to the case of the bending moment, the presence of constraint at the left end provides a mechanism for crack arrest with a decrease in the ERR value (at about 0.92\% of the strengthened region).\textsuperscript{24}

The effect of partitioning in the enriched elements is now investigated for the crack length of 400 mm. For the sake of comparison, \( 8 \times 8, \ 6 \times 6, \) and \( 4 \times 4 \) Gauss quadrature rules are used in all enriched elements. The results are depicted in Table 1. It is observed that using the results of partitioning method is more accurate than ordinary Gauss quadrature rules.

A parametric study for the value of strain ERRs is now presented by varying the CFRP plate thickness with all other parameters fixed. The results are illustrated in Figure 15 and compared with the analytical results obtained by Greco et al.,\textsuperscript{24} which show a good agreement. The analysis points out that an increase in plate thickness leads to an increase in strain ERRs.

The effect of using tip enrichments is examined by comparing the variations of stress components on the crack tip element with and without tip enrichments. Figure 16 compares the stress fields around the crack tip when \( a = 800 \text{ mm} \). Clearly, only using tip enrichment functions, the singular nature of stress field near the crack tip can be accurately represented.

**Simply supported FRP-reinforced beam under three-point bending crack stability and propagation**

This example is dedicated to the analysis of an interfacial crack between an FRP laminate and a concrete beam, previously studied by Greco et al.\textsuperscript{24} An interfacial crack at the edge of a simply supported beam, reinforced by FRP laminate and subjected to three-point bending is considered. The plane stress state is presumed. The geometry and boundary conditions of the problem are indicated in Figure 17 and the material properties are:

**Concrete.**

\[ E = 30,000 \text{ MPa}, \quad G_{12} = 12,820.53 \text{ MPa}. \]

![Figure 14. Variations of the strain ERRs with respect to crack length.](image)

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>XFEM</th>
<th>Without partitioning</th>
<th>Reference values</th>
</tr>
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<tbody>
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<td>With partitioning</td>
<td>Without partitioning</td>
<td></td>
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<tr>
<td>( \sigma )</td>
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<td>6 \times 6</td>
<td>4 \times 4</td>
</tr>
<tr>
<td>400</td>
<td>6.00</td>
<td>6.31</td>
<td>6.27</td>
</tr>
</tbody>
</table>
CFRP.

\[ E_1 = 160,000 \text{ MPa}, \quad E_2 = 9260 \text{ MPa}, \]
\[ G_{12} = 5333.3 \text{ MPa}, \quad v_{12} = 0.3. \]

The dimensions of the problem are:

\[ L_c = 3000 \text{ mm}, \quad L_p = 2400 \text{ mm}, \quad t_c = 300 \text{ mm}, \quad t_p = 4 \text{ mm}. \]

The FE mesh of Figure 18 is used to simulate the problem and the element which contains the crack tip is modeled with the new orthotropic enrichment functions. Again, the FRP laminate is discretized into 4.5 elements in \( y \)-direction. The total number of elements and nodes are 10,988 and 11,205, respectively.

Again, the same relative \( J \) integral domain size as example 2 is used to evaluate the strain ERRs of the present debonding problem.

Figure 19 illustrates the results of XFEM simulations for different crack lengths based on the new orthotropic enrichment functions.

Greco et al.\textsuperscript{24} performed a similar study on this problem. Figure 19 shows the results of analytical and numerical values of strain ERRs obtained by Greco et al.\textsuperscript{24} The numerical results were obtained by utilizing a mesh with the crack tip element size of 1 mm in the
x-direction and about 78,624 elements and 79,173 nodes.\textsuperscript{24}

It can be observed that the XFEM numerical estimates of strain ERRs are in close agreement with the analytical solution reported by Greco et al.\textsuperscript{24} It should be noted that these results are closer to analytical solution than the reference FE results while the number of elements in the XFEM mesh is about 1/7th of the reference FE mesh.\textsuperscript{24}

Moreover, a crack propagation problem is idealized in this problem to illustrate the efficiency of the present XFEM for simulation of several crack lengths on a fixed FE mesh. As a result of having the interlaminar crack to extend only along the material interface, no particular propagation criterion is necessary in examining this characteristic of present XFEM, and instead, different crack lengths are assumed to be generated through the propagation process. The results of ERR, which are evaluated on the fixed FE model for different crack lengths, are shown in Table 2, and compared with the previous results and the approximate reference solutions, showing very good quality results for all cases.

So far, the adhesive layer has been neglected and reasonably accurate results have been obtained. But this cannot be generalized to all problems. In some problems, it may be more feasible to separately simulate the adhesive layer (which can be performed by the present XFEM approach). In order to demonstrate such a capability and to verify the mentioned assumptions, this example is re-analyzed by a three-layer XFEM model, which includes FRP, the adhesive layer (for instance with the thickness of $t_a = 2$), and the concrete beam. The results, shown in Figure 19, demonstrate the little effect of inclusion of adhesive layer as a separate layer on the overall response for different crack lengths.

It should be re-emphasized, however, that such a conclusion cannot be generalized to all problems, and requires an independent study to assess the possible effects. It seems that a very thick adhesive layer is better to be simulated independent of the FRP laminate. This is the case for any potential adhesive layer with sufficient bending stiffness. On the contrary, thin adhesives can be simulated in combination with FRP as
a single layer or can even be neglected, while sufficiently accurate results are expected to be obtained for fracture parameters such as the ERR $G$.

**FRP-strengthened concrete beam**

To further assess the performance of the proposed approach, an FRP-strengthened concrete beam, previously studied by Rabinovitch\textsuperscript{26} is considered (Figure 20).

The strengthened beam is composed of concrete, an isotropic material, with the following properties:

**Concrete.**

\[ E = 30 \text{ GPa}, \nu = 0.3. \]

---

**Table 2.** Comparison of $GE_c t^2/f^2$ values for different crack lengths on a fixed mesh

<table>
<thead>
<tr>
<th>$a$ (mm)</th>
<th>XFEM Adapted mesh</th>
<th>XFEM Fixed mesh</th>
<th>Reference Analytical</th>
<th>Reference FEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>600</td>
<td>2.4295</td>
<td>2.4301</td>
<td>2.41</td>
<td>2.59</td>
</tr>
<tr>
<td>400</td>
<td>1.4654</td>
<td>1.5164</td>
<td>1.5</td>
<td>1.64</td>
</tr>
<tr>
<td>300</td>
<td>1.0835</td>
<td>1.1965</td>
<td>1.07</td>
<td>1.21</td>
</tr>
<tr>
<td>200</td>
<td>0.7928</td>
<td>0.8471</td>
<td>0.79</td>
<td>0.90</td>
</tr>
</tbody>
</table>

**Figure 20.** Geometry, loading, and boundary conditions.\textsuperscript{26}

**Figure 21.** Near-tip adaptively structured XFEM mesh for $a = 150$ mm.
and CFRP which is orthotropic with the following components of material modulus:

**CFRP.**

\[ E_1 = 158 \text{ GPa}, \quad E_2 = 32.7 \text{ GPa}, \quad G_{12} = 5.2 \text{ GPa}, \quad \nu_{12} = 0.27. \]

The crack is aligned along the interface of FRP strips and concrete beam. The values of strain ERRs are evaluated based on the adaptively structured mesh, as shown in Figure 21. The FRP laminate is discretized into 3.5 elements through the thickness. Also, the total number of elements and nodes in the FRP laminate are about 536 and 675, respectively.

Again a rectangular integration domain is utilized in order to evaluate the strain ERRs. The value of the relative J-integration domain size, \( r_d \) (the distance between the crack tip and each edge of the integration domain), is assumed to be about the thickness of CFRP strip below the crack edge and about 15 mm for other three edges.

To determine the accuracy of the approach, comparisons are made with the numerical investigation of Rabinovitch,\(^{26}\) which was based on the virtual crack extension method and using the high-order model and the J integral. The reference FE model, composed of bi-linear four node elements, was utilized with about 7000 quadrilateral elements in the CFRP strip with 8008 nodes.

Figure 22 represents the results of XFEM simulations as a function of crack length, which are closely comparable with the results of strain ERRs obtained by the J integral method in Rabinovitch\(^{26}\) report.

Table 3 indicates the results of strain ERR by changing the size of relative integration domain, \( r_d \) (the distance between the crack tip and each edge of the integration domain). It is assumed that \( r_d \) is about the thickness of CFRP strip below the crack edge and the other three edges differ.

The results indicate that the overall differences between XFEM and the reference results vary between 0.05% and 5.5%. Since the reference results cannot be regarded as the accurate and exact solution, while a general conclusion cannot be made to optimally assign a particular \( r_d \) for each crack length in this example, any values of \( r_d \) in Table 3 can be used to obtain an acceptable engineering result. It should be noted that the path independency of the interaction integral is only a pure analytical hypothesis, which may slightly be violated in numerical analyses due to their own extra assumptions and approximations.

**Delamination of metallic I beams strengthened by FRP strips**

In this example, a straight simply supported I shaped steel beam with CFRP reinforcement subjected to a point loading, as shown in Figure 23, is studied. Two debonding cracks are located symmetrically at the interface of the steel beam and the CFRP strips at the two ends (Figure 23). Due to the reinforcement symmetry, the analysis is performed for half of the beam in a plane stress state for a range of crack lengths.

![Figure 22. Strain ERRs vs. crack length curves.](image)

**Table 3.** The effect of different sizes of the relative J integral domain \((r_d)\) on the values of ERR for different crack lengths

<table>
<thead>
<tr>
<th>( a ) (mm)</th>
<th>( r_d )</th>
<th>( \sigma ) (( \text{J/m}^2 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>10</td>
<td>89.13</td>
</tr>
<tr>
<td>100</td>
<td>12</td>
<td>90.21</td>
</tr>
<tr>
<td>200</td>
<td>15</td>
<td>90.96</td>
</tr>
<tr>
<td>50</td>
<td>17</td>
<td>90.95</td>
</tr>
<tr>
<td>100</td>
<td>20</td>
<td>91.50</td>
</tr>
<tr>
<td>200</td>
<td>20</td>
<td>90.91</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \sigma ) (mm)</th>
<th>Reference results(^{26})</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.76</td>
</tr>
<tr>
<td>2.6</td>
<td>1.5</td>
</tr>
<tr>
<td>3.1</td>
<td>4.2</td>
</tr>
</tbody>
</table>

\( \text{Relative differences from the reference values} \ (\%) \)

<table>
<thead>
<tr>
<th>( \sigma ) (mm)</th>
<th>( 2 )</th>
<th>( 2.6 )</th>
<th>( 3.1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.05</td>
<td>0.7</td>
<td>5</td>
</tr>
<tr>
<td>12</td>
<td>0.05</td>
<td>0.7</td>
<td>5</td>
</tr>
<tr>
<td>15</td>
<td>0.6</td>
<td>0.17</td>
<td>5.5</td>
</tr>
</tbody>
</table>
The results of the present simulation are compared with the results of Colombi. Material properties are defined as:

**Steel.**

\[ E = 210 \text{ GPa}, \quad \nu_{12} = 0.3. \]

**CFRP strips.**

\[ E_1 = 197 \text{ GPa}, \quad E_2 = 15.8 \text{ GPa}, \]
\[ G_{12} = 8.18 \text{ GPa}, \quad \nu_{12} = 0.3. \]

Adaptive structured FE meshes with different number of quadrilateral four-noded elements for different crack lengths \( (a = 5 - 160 \text{ mm}) \) are used in this example. The mesh is fixed and composed of 266 quadrilateral elements and 402 nodes in the CFRP strip.

Figure 24 illustrates the typical mesh utilized for \( a = 100 \text{ mm} \).

In these models, element sizes are substantially smaller in the vicinity of the crack, while the FRP laminate is discretized into 1.5 elements in \( y \)-direction. The total number of rows and columns of FEs are 81 and 134, respectively.

The value of the relative \( J \)-integration domain size, \( r_d \) (the distance between the crack tip and each edge of the integration domain), is assumed to be about the thickness of CFRP strip below the crack edge and about 25 mm for other three edges.

The results of the XFEM simulations for different values of the initial delamination length are compared with the results of Colombi’s study. Colombi evaluated the strain ERRs using both analytical and numerical solutions. His results from the transformed section approach were evaluated analytically, while the
Figure 25. Details of the reference two-dimensional FE model of the reinforced beam.\textsuperscript{29}

![Diagram showing details of the reference two-dimensional FE model of the reinforced beam.]

Figure 26. Strain ERR vs. the delamination crack length.

![Graph showing strain ERR vs. delamination crack length with data points and lines for XFEM, Trans. Sec. [29], Elast. Found. [29], and FEM [29].]
ERR from the two-parameter elastic foundation model was evaluated numerically using a virtual crack length equal to 1 mm. The model was constructed by a one-dimensional FE for the beam and a two-dimensional shell FE model was adopted for the composite strips and adhesive layer, as depicted in Figure 25. The number of elements and nodes in the CFRP strip were about 3960 and 4955, respectively.

The XFEM results and reference results are illustrated in Figure 26. Comparisons between the results of all three methods indicate a very good agreement between the proposed model and the reference results.

**Variable section beam reinforced by FRP**

In this example, a cantilever, variable section beam is subjected to an edge transverse force \( F = 30 \text{ N} \). The geometry of the problem is shown in Figure 27. The beam is composed of concrete and reinforced with orthotropic CFRP plates. The material properties are defined as:

**Concrete.**

\[ E = 30,000 \text{ MPa}, \ G_{12} = 12,820.53 \text{ MPa}. \]

**CFRP strips.**

\[ E_1 = 160,000 \text{ MPa}, \ E_2 = 7610 \text{ MPa}, \ G_{12} = 5333.3 \text{ MPa}, \ \nu_{12} = 0.3. \]

The XFEM analysis is performed for a range of crack lengths in a plane stress state in order to evaluate the strain ERRs. An unstructured FE model, depicted in Figure 28, is used for different crack lengths which
Figure 30. X and Y displacements and $\sigma_{xx}$, $\sigma_{yy}$, and $\sigma_{xy}$ contours.
represent a crack propagation problem. The crack tip is located in the finer mesh area and modeled with the proposed orthotropic enrichment functions. The mesh discretization is composed of 1152 four-noded quadrilateral elements and 1232 nodes. This is in fact a test to demonstrate the ability of the proposed approach with rather complex debonding problems.

Figure 29 depicts the results of \( G_T E_t t_2 / F^2 \) obtained by XFEM analysis for different crack lengths. A rapid drop in \( G_T \) is predicted at \( a = 1100 \text{ mm} \), where the interlaminar crack extends to the vicinity of clamped edge.

Displacement and stress contours of the present example for \( a = 900 \text{ mm} \) are plotted in Figure 30.

**Conclusions**

An XFEM model for debonding failure analysis of beams strengthened with externally bonded FRP plates is proposed. Bimaterial orthotropic enrichment functions, the Heaviside function, and weak discontinuity enrichment functions are utilized in XFEM analysis to simulate the interface crack between two linear elastic isotropic (beam) and orthotropic (FRP plate) materials and fracture analysis of the domain. Combined modes I and II loading conditions are considered and the mixed-mode SIFs numerically evaluated using the domain interaction integral approach. The strain ERR is evaluated based on the SIFs. Results obtained by the XFEM approach compared with predictions from various reference investigations outline the capability of the proposed model to predict the debonding failure behaviour. The computed ERRs exhibit a satisfactory estimate with the available reference results. In addition to simulation of several crack lengths (as a representation of crack propagation) on a fixed FE mesh, effect of different types and sizes of mesh and different relative \( J \) integral domain sizes has been studied comprehensively to investigate the efficiency and robustness of the present method.

The present XFEM approach is capable of modeling any number of layers. The necessity of modeling the adhesive layer depends mainly on the general modeling concepts, the required accuracy, and the computational costs. While we may choose to model the adhesive layer as a separate layer or an equivalent layer of FRP + adhesive, the obtained results demonstrate sufficient accuracy even if the adhesive layer is neglected. It should be emphasized, however, that such a conclusion cannot be generalized to all problems; as it may be more feasible to separately simulate the adhesive layer (which can be performed by the present XFEM approach). This requires an independent study to assess the possible effects. It seems that a very thick adhesive layer is better to be simulated independent of the FRP laminate. This is the case for any potential adhesive layer with sufficient bending stiffness. On the contrary, thin adhesives can be simulated in combination with FRP as a single layer or can even be neglected, while sufficiently accurate results are expected to be obtained for fracture parameters such as the ERR \( G \).

**Acknowledgments**

The authors acknowledge the financial support of University of Tehran for this research under the grant number 8102051/1/02. Also, the technical support of the High Performance Computing Lab, School of Civil Engineering, University of Tehran is gratefully acknowledged.

**References**

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Appendix

The general crack tip displacement fields for a crack along the interface of two dissimilar orthotropic materials under the mixed-mode loading have been obtained by Lee\textsuperscript{56} (also see Ref.\textsuperscript{45}).

It should be noted that the following displacement fields are for the upper material and in order to obtain the displacement fields in the material below the interface, parameters $\varepsilon$ and $-\varepsilon$ are changed to $-\varepsilon$ and $\varepsilon$, respectively.

\begin{align}
  u_e & = \frac{K_I}{\sqrt{2\pi}(1 + 4\varepsilon^2)D} \left\{ e^{i(\pi - \theta)} \hat{p}_A \left[ \cos \left( \varepsilon \ln(r) - \frac{\theta}{2} \right) + 2\varepsilon \sin \left( \varepsilon \ln(r) + \frac{\theta}{2} \right) \right] 
  \right. \\
  & \quad \left. + \varepsilon e^{i(\pi - \theta)} \hat{p}_A \left[ \cos \left( \varepsilon \ln(r) + \frac{\theta}{2} \right) + 2\varepsilon \sin \left( \varepsilon \ln(r) - \frac{\theta}{2} \right) \right] (r)^2 \right\} \\
  & \quad + \frac{K_{II}}{\sqrt{2\pi}(1 + 4\varepsilon^2)D} \left\{ -\varepsilon e^{i(\pi - \theta)} \hat{p}_A \left[ \sin \left( \varepsilon \ln(r) + \frac{\theta}{2} \right) - 2\varepsilon \cos \left( \varepsilon \ln(r) + \frac{\theta}{2} \right) \right] 
  \right. \\
  & \quad \left. + \varepsilon e^{i(\pi - \theta)} \hat{p}_A \left[ \sin \left( \varepsilon \ln(r) - \frac{\theta}{2} \right) - 2\varepsilon \cos \left( \varepsilon \ln(r) - \frac{\theta}{2} \right) \right] (r)^2 \right\} \\
  & \quad + \varepsilon^2 e^{i(\pi - \theta)} \hat{p}_A \left[ \sin \left( \varepsilon \ln(r) - \frac{\theta}{2} \right) - 2\varepsilon \cos \left( \varepsilon \ln(r) - \frac{\theta}{2} \right) \right] (r)^2 \\
  & \quad \left. + \varepsilon^2 e^{i(\pi - \theta)} \hat{p}_A \left[ \sin \left( \varepsilon \ln(r) + \frac{\theta}{2} \right) - 2\varepsilon \cos \left( \varepsilon \ln(r) + \frac{\theta}{2} \right) \right] (r)^2 \right\}
\end{align}
where

\[ p_l = -q^2a_{11} + a_{12}, \quad p_s = -q^2a_{11} + a_{12}, \quad q_l = \frac{-p^2a_{12} + a_{22}}{p}, \quad q_s = \frac{-q^2a_{12} + a_{22}}{q} \quad (A3) \]

and

\[ A = q + \eta, \quad \tilde{A} = q - \eta, \quad B = p + \eta, \quad \tilde{B} = p - \eta, \quad D_k = [q - p]_k, \quad \eta = \left( \frac{h_{13}}{h_{12}} \right)^{1/2} \quad (A4) \]

\[ \begin{cases} h_{11} = (l_{11})_1 - (l_{11})_2 \\ h_{21} = (l_{12})_1 + (l_{12})_2 \\ h_{12} = (l_{21})_1 + (l_{21})_2 \end{cases} \quad (A5) \]

\[ \begin{cases} (l_{11})_k = \left( \frac{p-p-q}{q-p} \right)_k = \left( \frac{q-q-p}{q-p} \right)_k \\ (l_{12})_k = \left( \frac{p-p-q}{q-p} \right)_k \\ (l_{21})_k = \left( \frac{q-q-p}{q-p} \right)_k \end{cases} \quad (A6) \]

Subscripts \( k = 1, 2 \) denote the upper and lower materials, respectively. Also, variables \( r_z, \theta_z(z = l, \tilde{l}) \) are related to polar coordinate system \((r, \theta)\) as

\[ \begin{cases} r_z = r \sqrt{\cos^2 \theta + Z_2^2 \sin^2 \theta}, \quad Z_2 = p \\ \theta_z = \tan^{-1}(Z_2 \tan \theta), \quad Z_3 = q \end{cases} \quad (A7) \]

\( K_I \) and \( K_H \) are the modes 1 and 2 SIFs, respectively, and \( \varepsilon \) is the index of oscillation defined as

\[ \varepsilon = \frac{1}{2\pi} \ln \left( \frac{1 - \beta}{1 + \beta} \right) \quad (A8) \]

where \( \beta \) is the second Dundurs parameter\(^{56} \) given by

\[ \beta = \frac{h_{11}}{\sqrt{h_{12}h_{21}}} \quad (A9) \]