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# Enhancement of Isogeometric Analysis Method for Analyzing 2D Cracked Problems Using Extrinsic Enrichment Functions

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**Abstract:** Isogeometric Analysis (IGA) is a robust computational approach with highly precise geometry representation and has attracted rapidly growing research interests in a wide range of applications. It opens up the possibility integrating the Finite Element Analysis (FEA) into the Computer Aided Design (CAD) tools. Non-Uniform Rational B-spline (NURBS) functions of IGA make it possible to model and analyze complex problems with the least points and DOFs. In this study, the method is further improved for analysis of discontinuous problems using the efficient concepts of the Extended Finite Element Method (XFEM). XFEM allows for the entire crack to be represented independent of the mesh and is very efficient in crack propagation simulations, because the element boundaries need not to be aligned to crack surfaces and therefore, no remeshing is necessary. Also, the accuracy of solution is improved by enhancing the description of singular stress fields near the crack tip through the crack tip enrichment functions. In this contribution, the Heaviside and isotropic crack tip enrichment functions are applied in the IGA method for accurate simulation of crack propagation problems. Several 2D linear elastic fracture problems are numerically simulated in static and quasi-static conditions to demonstrate the suitability of the proposed approach for modeling discontinuous problems.

## 1. Introduction

Analysis of a cracked body in an isotropic medium without the necessity of remeshing, applying an efficient numerical method is the aim of this contribution. Since the analytical methods are not capable of solving many complicated engineering problems, numerical methods have been developing increasingly regarding to the fast development of computers. One of latest developed method is the isogeometric analysis (IGA) (Hughes et al. 2005) which applies the NURBS function for both basis function and geometry description and enables complex geometries to be modeled with few points. This method has been effectively and growingly applied in a number of engineering problems (Cottrell et al. 2009), including damage and fracture mechanics (Verhoosel et al. 2010a, 2010b). Verhoosel et al. (2010b) used the knot insertion technique and discretized the cohesive zone formulation to analyze cohesive cracks by the IGA, but the reparametrization process in each step of crack growth is unavoidable.

Omission of remeshing necessity in moving discontinuous problems for decreasing the computational costs and human labors, has motivated researchers to develop extended finite element method (XFEM) (Belytschko et al. 1999), mesh-free methods (Liu 2003), etc. In this paper, the appropriate concepts of XFEM are used in the IGA so that control points are extrinsically enriched with Heaviside and crack tip enrichment functions. The new approach which is called extended isogeometric analysis (XIGA), benefits from the advantages of both the XFEM and the IGA. The XIGA can analyze mixed mode crack propagation problems without the need of reparametrization process.

The framework of isogeometric analysis is firstly discussed in Section 2. Then in Section 3, crack modeling and applying extrinsic enrichment functions in IGA are described and formulation of the proposed approach is presented. Subsequently, several numerical simulations are illustrated in Section 4 to demonstrate the efficiency and robustness of the present method in analysis of crack stability and propagation. Finally, concluding remarks are presented in Section 5.

## 2. Isogeometric analysis

As a prelude to description of solution field approximation for the isogeometric analysis, the B-spline and NURBS functions are briefly discussed in this section. For more details, refer to (Piegl et al. 1997).

A non-uniform rational B-spline (NURBS) curve of order $p$ is defined as

[1] $$\mathbf{C}(\xi)=\sum_{i=1}^{n} R_i^p(\xi)\mathbf{T}_i \qquad 0\le\xi\le 1$$

where $\{\mathbf{T}_i\}=\left(X_{i_1},X_{i_2}\right)$ are the coordinate position of a set of $i=1,2,\ldots,n$ control points and $\left\{R_i^p\right\}$ are the NURBS functions which are defined as follow

[2] $$R_i^p(\xi)=\frac{N_{i,p}(\xi)w_i}{\sum_{\hat{i}=1}^{n} N_{\hat{i},p}(\xi)w_{\hat{i}}}$$

where $\{w_i\}$ are weights corresponding to the control points and $N_{i,p}$ are the B-spline basis functions of order $p$ which are defined in a parametric space of the so-called knot vector $\boldsymbol{\Xi}$.

[3] $$\boldsymbol{\Xi}=\left\{\xi_1,\xi_2,\ldots,\xi_{n+p+1}\right\} \qquad \xi_i\le\xi_{i+1},\ i=1,2,\ldots,n+p$$

where the knots $\{\xi_i\}$, $(i=1,2,\ldots,n+p+1)$ are real numbers representing the coordinates in the parametric space $[0,1]$. In the isogeometric analysis, with the purpose of satisfying the Kronecker delta property at boundary points, an open knot vector (a knot vector with p+1 repeated knots at the ends) is used (Roh et al. 2004).

B-spline basis functions are defined in the following recursive form (Piegl et al. 1997)

[4] $$N_{i,0}(\xi)=\begin{cases}1 & \text{if } \xi_i\le\xi\le\xi_{i+1}\\ 0 & \text{otherwise}\end{cases}$$

[5] $$N_{i,p}(\xi)=\frac{\xi-\xi_i}{\xi_{i+p}-\xi_i}N_{i,p-1}(\xi)+\frac{\xi_{i+p+1}-\xi}{\xi_{i+p+1}-\xi_{i+1}}N_{i+1,p-1}(\xi) \quad \text{for } p=1,2,3,\ldots$$

A NURBS surface of order $p$ in $\xi_1$ direction and order $q$ in $\xi_2$ direction is defined in the form of

$$[6] \quad \mathbf{S}(\xi_1,\xi_2)=\sum_{i=1}^{n}\sum_{j=1}^{m}\underbrace{\frac{N_{i,p}(\xi_1)M_{j,q}(\xi_2)w_{i,j}}{\sum_{\hat{i}=1}^{n}\sum_{\hat{j}=1}^{m}N_{\hat{i},p}(\xi_1)M_{\hat{j},q}(\xi_2)w_{\hat{i},\hat{j}}}}_{R_{i,j}^{p,q}(\xi_1,\xi_2)}\mathbf{T}_{i,j} \qquad 0\le\xi_1,\xi_2\le 1$$

where $\{w_{i,j}\}$ are the weights, $\{\mathbf{T}_{i,j}\}$ represent a $n \times m$ bidirectional control net, and $\{N_{i,p}\}$ and $\{M_{j,q}\}$ are the B-spline basis functions defined on the knot vectors $\Xi_1$ and $\Xi_2$, respectively.

Formulation of isogeometric analysis is the same as finite element method with the difference that instead of classical Lagrangian basis functions, NURBS functions are used for both the parameterization of the geometry and the approximation of the solution field $\mathbf{u}^h$. The physical coordinates $\mathbf{X}=(X_1,X_2)$ an displacement approximation $\mathbf{u}^h$ for a typical point $\boldsymbol{\xi}=(\xi_1,\xi_2)$ (parametric coordinates) are computed according to the Eqs. (7) and (8), respectively.

$$[7] \quad \mathbf{X}(\boldsymbol{\xi})=\sum_{i=1}^{n_{cp}}R_i(\boldsymbol{\xi})\mathbf{T}_i$$

$$[8] \quad \mathbf{u}^h(\boldsymbol{\xi})=\sum_{i=1}^{n_{cp}}R_i(\boldsymbol{\xi})\mathbf{u}_i$$

where $n_{cp}$ is the number of control points and $\mathbf{u}_i$ are called control variables. It is noted that a control point does not generally coincide with an element vertex in the physical space.

## 3. Extended isogeometric analysis

The idea of Extended Isogeometric Analysis (XIGA) is to use a classical IGA space enriched by some additional functions applying the superior concepts of the Extended Finite Element Method (XFEM) (Belytschko et al. 1999, Mohammadi 2008). These functions result from the product of global enrichment functions and some classical NURBS functions. In this case, discontinuous problems can be efficiently analyzed so that remeshing necessity is vanished in moving discontinuous problems, such as crack propagation simulations. In the present method, some control points are selected to be enriched by Heaviside and crack tip enrichment functions for modeling crack face and improving the accuracy of solution field near the crack tip, respectively. So degrees of freedom are increased according to the number of these enriched control points.

### 3.1 Basic equations

For modeling crack edges and tips in XIGA, the standard IGA approximation (Eq. (8)) is enriched in the following form

[9] $\mathbf{u}^h(\xi)=\sum_{i=1}^{n_{cp}} R_i(\xi)\mathbf{u}_i+\sum_{j=1}^{n_{cf}} R_j(\xi)H(\xi)\mathbf{b}_j+\sum_{k=1}^{n_{ct}} R_k(\xi)\left(\sum_{\alpha=1}^{4} Q_\alpha(\xi)\mathbf{c}_k\right)$

where $\mathbf{b}_j$ is the vector of additional degrees of freedom which are related to the modeling of crack faces (not crack tips), $\mathbf{c}_k$ is the vector of additional degrees of freedom for modeling the crack tip, $n_{cf}$ is the number of basis functions that have crack face (but not crack tip) in their support domain and $n_{ct}$ is the number of basis functions associated with the crack tip in their influence domain. $H(\xi)$ is the generalized Heaviside function (Moës et al. 1999), which becomes +1 if $\mathbf{X}$ (physical coordinates corresponding to the parametric coordinates $\xi$) is above the crack and –1, otherwise and $Q_\alpha$, $(\alpha=1,2,3,4)$ are crack tip enrichment functions which are defined as the following equation for the isotropic materials

[10] $Q_\alpha(r,\theta)=\{Q_1,Q_2,Q_3,Q_4\}=\left\{\sqrt{r}\sin\frac{\theta}{2},\sqrt{r}\cos\frac{\theta}{2},\sqrt{r}\sin\theta\sin\frac{\theta}{2},\sqrt{r}\sin\theta\cos\frac{\theta}{2}\right\}$

$(r,\theta)$ are the local polar coordinates at the crack tip.

As mentioned before, Eq. (9) defines the new extrinsically enriched displacement approximation for a typical point $\xi$ in the proposed XIGA approach. The first term in the right-hand side of Eq. (9) is the classical IGA approximation to determine the displacement field, while the remaining terms are the enrichment approximation in order to model discontinuity and to accurately represent the analytical solution near the crack tip.

### 3.2 Enriched control points

In the present method, as schematically illustrated in Figure 1, the control points which the support domain of their corresponding basis functions contain the crack tip should be enriched with crack tip enrichment functions and the control points which influence domains of their corresponding basis functions intersect with the crack face (not crack tip) are enriched with the Heaviside function.

It is shown This is schematically illustrated in Figure 1, where the control points $\mathbf{T}_j$ and $\mathbf{T}_k$ are selected as the sample crack face and crack tip enriched control points. Selection of enriched control points is performed as follows: Firstly, NURBS functions of the crack tip point are obtained. The non-zero NURBS values $R_i(\xi_{tip})\neq 0$, $i=1,2,\ldots,n_{cp}$ specify the crack tip enriched control points ($R_k(\xi_{tip})\neq 0$ in Figure 1). The same procedure can be applied to select the Heaviside enriched control points with the difference that the crack tip is replaced by some points on the crack face ($\mathbf{X}_{c1},\mathbf{X}_{c2},\ldots,\mathbf{X}_{c7}$ instead of $\mathbf{X}_{tip}$ in Figure 1). It is noted that the control points selected for both Heaviside and crack tip enrichments, are only considered as crack tip enriched control points. Accordingly, despite the fact that $R_k(\xi_{tip})\neq 0$, $R_k(\xi_{c7})\neq 0$ for the control point $\mathbf{T}_k$ in Figure 1, it is selected only as a crack tip enriched control point.

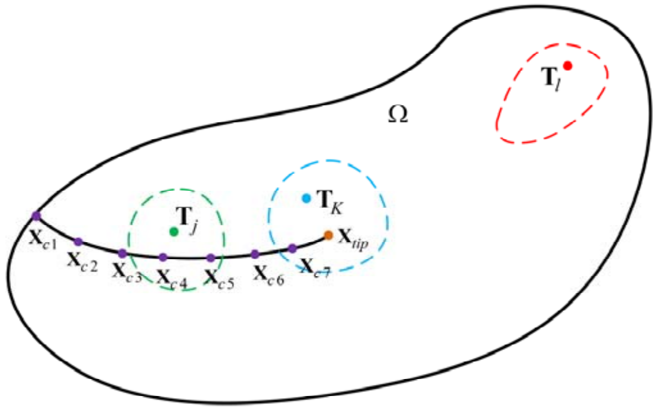


Figure 1: Selection of enriched control points: $\mathbf{T}_j$ is a Heaviside enriched control point because of $R_j(\xi_{c4}), R_j(\xi_{c5}) \neq 0$, whereas $\mathbf{T}_k$ is a crack tip enriched control point since $R_k(\xi_{c7}), R_k(\xi_{tip}) \neq 0$, . $\mathbf{T}_l$ is assumed an ordinary control point, because $R_l(\xi_{ci}) = 0$, $i = 1, 2, \ldots, 7$ and $R_l(\xi_{tip}) = 0$.

### 3.3 Sub-triangles technique

The Gauss quadrature rule is utilized for integration over the XIGA elements. Existence of discontinuity within an element may result in substantial accuracy reduction. Therefore, an efficient technique is required to define the necessary points needed for the integration within these elements, while remains consistent with the crack geometry.

In order to overcome this numerical difficulty, an approach called sub-triangles technique, originated from XFEM (Dolbow et al. 1999), and similar to the one proposed by Ghorashi et al. (2011) for element free Galerkin method (EFGM) is adopted. According to this technique, any element which intersects with a crack is subdivided at both sides into sub-triangles whose edges are adjusted to crack edges, as illustrated in Figure 2.

It is worth noting that, while triangulation of the crack tip element substantially improves the accuracy of integration by increasing the order of Gauss quadrature, it also avoids numerical complications of singular fields at the crack tip because none of the Gauss points are placed on the position of the crack tip.

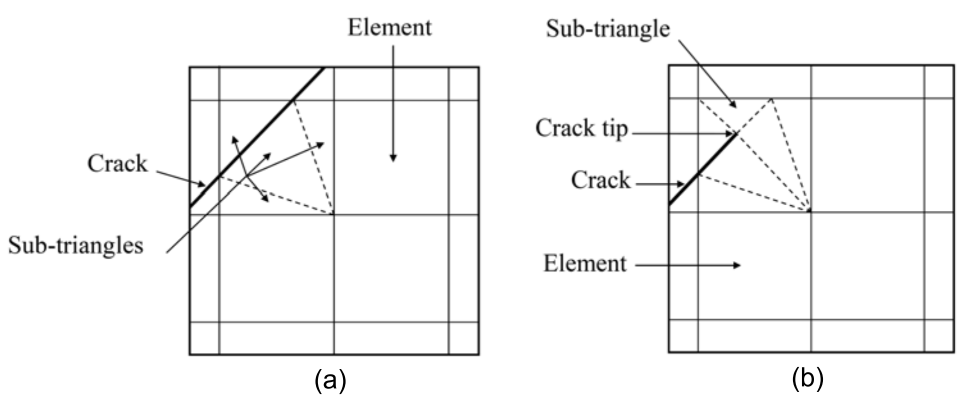


Figure 2: The sub-triangles technique for partitioning the cracked elements (Ghorashi et al. 2011): (a) crack face; and (b) crack tip.

## 4. Numerical examples

In this section, we present the evaluations of numerical results in the linear fracture mechanics to confirm the usefulness and efficiency of the XIGA including comparisons with that by using the XFEM. Following two types of models including cracks will be used; a Mode I crack model in an infinite plate, and crack propagation in a double cantilever beam.

In the examples, all knot vectors are open and uniform without any interior repetition the weights are assumed to be unity, the order of NURBS functions in both parametric directions $\xi_1$ and $\xi_2$ are considered three ( $p = q = 3$ ) and for integration, 4×4 Gauss quadrature and sub-triangles technique with 13 Gauss points in each sub-triangle are used. The Lagrange multiplier method (Belytschko et al. 1994) has been adopted for imposition of essential boundary conditions.

Stress intensity factors are obtained using the interaction integral method for determination of fracture properties.

### 4.1 Mode I crack model in the infinite plate

An infinite plate including a straight crack under pure fracture mode I is considered and depicted in Figure 3. The plate is in plane strain state. Then, let us define a local finite square domain ABCD including the crack tip in the center. The domain ABCD, which includes the $c_l = 5\,\mathrm{mm}$ part of the crack, is smaller than the crack length $2a = 200\,\mathrm{mm}$ in infinite plate. The size of this analytical domain ABCD is 10×10 mm. Other parameters are:

$$E = 10^7\ \mathrm{N/mm^2},\ \ \nu = 0.3,\ \ \sigma_o = 10^4\ \mathrm{N/mm^2}$$

The analytical solution for the displacement and stress fields in terms of local polar coordinates in a reference frame $(r,\theta)$ centered at the crack tip are:

[11]
$$\begin{aligned} u_x(r,\theta) &= \frac{2(1+\nu)}{\sqrt{2\pi}}\frac{K_I}{E}\sqrt{r}\cos\frac{\theta}{2}\left(2-2\nu-\cos^2\frac{\theta}{2}\right) \\ u_y(r,\theta) &= \frac{2(1+\nu)}{\sqrt{2\pi}}\frac{K_I}{E}\sqrt{r}\sin\frac{\theta}{2}\left(2-2\nu-\cos^2\frac{\theta}{2}\right) \end{aligned}$$

[12]
$$\begin{aligned} \sigma_{xx}(r,\theta) &= \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \\ \sigma_{yy}(r,\theta) &= \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \\ \sigma_{xy}(r,\theta) &= \frac{K_I}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} \end{aligned}$$

Analytical displacement field (Equations 11) are prescribed on bottom, right and top edges, while prescribed forces (Equations 121) are applied on the left edge.

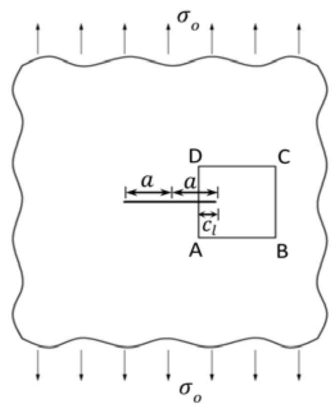


Figure 3: An mode I crack model in an infinite plate.

Several control meshes (36 to 1444 control points) are modeled for investigating the accuracy and convergence rate of the XIGA. For better evaluation of the proposed approach, this problem is also analyzed by XFEM (with the same number of nodes) and the results of both methods are compared with the exact solution. Figure 4 illustrated the $L_2$ error norms of results for different number of points (control points in XIGA or nodes in XFEM). It is shown that XIGA has higher accuracy and convergence rate in comparison with XFEM. It is worth noting that, $L_2$ error norm for model of 144 control points with 81 elements and 516 DOFs in the proposed method is 0.0809 %, while XFEM with 1444 nodes, 1369 elements and 2992 DOFs results in $L_2$ error norm of 0.1354 %. Figure 4 distinguishes these cases by dashed circles.

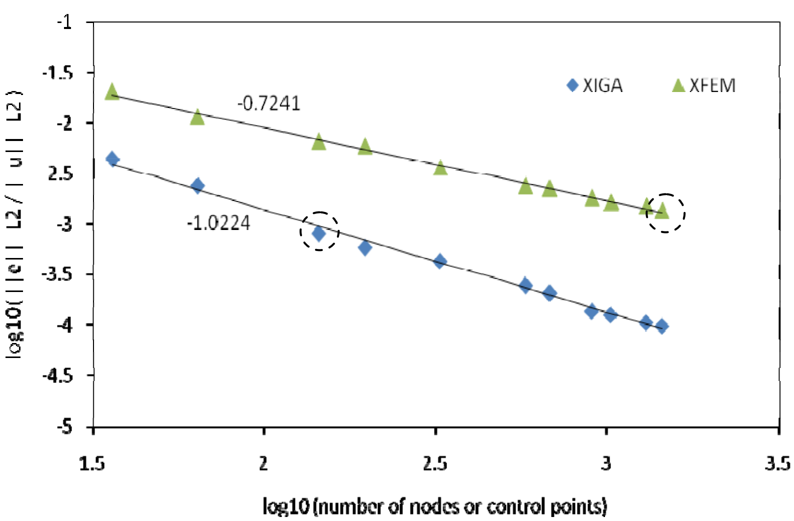


Figure 4: Convergence of $L_2$ error norm of XIGA and XFEM.

Figure 5 shows the control points distribution and the Heaviside and crack tip enriched control points for the model of 144 control points in the analysis of XIGA.

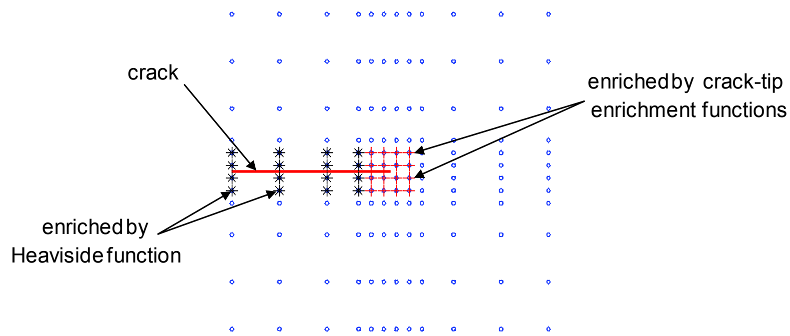


Figure 5: Distribution of control points (red cross signs represent the control points enriched by crack tip enrichment functions and black star signs represent the control points enriched by the Heaviside function).

### 4.2 Crack propagation in a double cantilever beam

A more challenging example is to investigate the crack propagation under mixed-mode loading. The specimen and its geometry are depicted in Figure 6. The initial crack propagates in the double cantilever beam in plane stress state. Applied parameters are:

$$P = 1.0\times10^5 \text{ N}, \quad E = 3.0\times10^7 \text{ N/mm}^2, \qquad \nu = 0.3$$

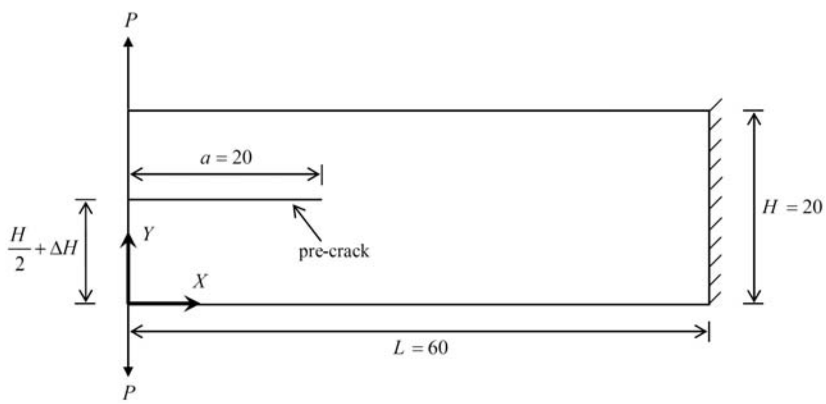


Figure 6: Geometry and loading of the double cantilever beam.

It is obvious that a crack placed exactly in the mid-plane, grows straightly under pure mode I according to symmetry, while if the position of initial crack be slightly out of the mid-plane, it will propagate in a curved path outwards of the center. A pre-crack with length of $a = 20$ mm is located slightly off the mid-plane ( $\Delta H$ ).

Quasi-static crack propagation is governed by the following maximum circumferential stress criterion (Erdogan et al. 1963) and the crack length increment is $\Delta a = 1$ mm is used.

[13] $\theta_c = 2\tan^{-1}\frac{1}{4}\left(\frac{K_I}{K_{II}} \pm \sqrt{\left(\frac{K_I}{K_{II}}\right)^2 + 8}\right)$

where $\theta_c$ is the angle of crack growth in crack tip local coordinate system. The mixed mode SIFs are computed using the domain form of the interaction integral in a circular domain with radius $r_J = 0.15 \times a$.

Crack propagation is simulated by modeling 63×24 control points with unit weights and 60×21 elements and analyzing in 11 steps. The influence of eccentricity $\Delta H$ on the crack propagation path is investigated. Figure 7 demonstrates the crack propagation paths for different values of initial $\Delta H = 0,\ 0.015,\ 0.035,\ 0.07,\ 0.14,\ 0.3,\ 0.5,\ 1\ \mathrm{mm}$. It is observed that when the initial crack locates in mid-plane ($\Delta H = 0$), it propagates only along the mid-plane ($Y = 10\ \mathrm{mm}$) which was expected.

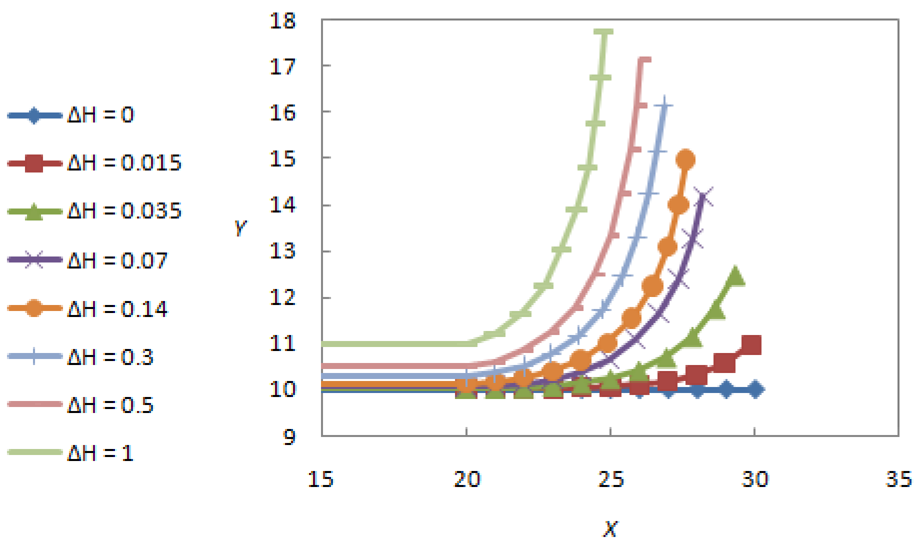


Figure 7: Influence of initial crack distance from the mid-plane (ΔH) on the crack paths in a double cantilever beam specimen.

## 5. Concluding remarks

In this paper, the conventional isogeometric analysis has been extended to analysis of discontinuous problems applying the enrichment functions applied in the extended finite element method. The new approach called Extended Isogeometric Analysis (XIGA) includes the advantage of both IGA and XFEM. Crack propagation simulations can be performed without the need of remeshing process. Moreover, the the sub-triangles technique has been adopted in the proposed approach to significantly improve the Gauss quadrature rule near the crack face and avoiding numerical complications of singular fields at the crack tip. Robustness and accuracy of XIGA have been confirmed by static and quasi-static illustrative examples.

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